மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

MANONMANIAM SUNDARANAR UNIVERSITY

TIRUNELVELI-627 012

தொலைநிலை தொடர் கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

M.Sc. MATHEMATICS

I YEAR

REAL ANALYSIS-II

Sub. Code: SMAM22

Prepared by

S. Kalaiselvi

Assistant Professor

Department of Mathematics

Sarah Tucker College(Autonomous), Tirunelvei-7.

M.Sc. MATHEMATICS –I YEAR

SMAM22: REAL ANALYSIS-II

SYLLABUS

Unit I

Measure on the Real line - Lebesgue Outer Measure - Measurable sets - Regularity - Measurable Functions - Borel and Lebesgue Measurability.

Chapter - 1 Sec 1.1 - 1.5

Unit II

Integration of Functions of a Real variable - Integration of Non- negative functions - The General Integral - Riemann and Lebesgue Integrals.

Chapter - 2 Sec 2.1-2.3

UNIT III

Fourier Series and Fourier Integrals - Introduction - Orthogonal system of functions - The theorem on best approximation - The Fourier series of a function relative to an orthonormal system - Properties of Fourier Coefficients - The Riesz-Fischer Theorem - The convergence and representation problems in for trigonometric series - The Riemann - Lebesgue Lemma - The Dirichlet Integrals - An integral representation for the partial sums of Fourier series - Riemann's localization theorem - Sufficient conditions for convergence of a Fourier series at a particular point –Cesaro Summability of Fourier series- Consequences of Fejer's theorem - The Weierstrass approximation theorem

Chapter 3: Sections 3.1 to 3.14

Unit IV

Multivariable Differential Calculus - Introduction - The Directional derivative - Directional derivative and continuity - The total derivative - The total derivative expressed in terms of partial derivatives - The matrix of linear function - The Jacobian matrix - The chain rule - Matrix form of chain rule - The mean - value theorem for differentiable functions - A sufficient condition for differentiability - A sufficient condition for equality of mixed partial derivatives - Taylor's theorem for functions of R^n to R^1

Chapter 4: Section 4.1 to 4.14

Unit V

Implicit Functions and Extremum Problems: Functions with non-zero Jacobian determinants – The inverse function theorem-The Implicit function theorem -Extrema of real valued functions of severable variables -Extremum problems with side conditions.

Chapter 5: Sections 5.1 to 5.7

Recommended Text:

1. G. de Barra, *Measure Theory and Integration*, Wiley Eastern Ltd., New Delhi, 1981. (for Units I and II)
2. Tom M. Apostol : *Mathematical Analysis*, 2^{nd} Edition, Addison-Wesley Publishing Company Inc. New York, 1974. (for Units III, IV and V)

SMAM22: REAL ANALYSIS-II

CONTENTS

UNIT-I

UNIT-II

UNIT-III

Unit-I

Measure on the Real line - Lebesgue Outer Measure - Measurable sets - Regularity - Measurable Functions - Borel and Lebesgue Measurability.

Chapter – 1 Sec 1.1 to 1.5

Measure on the Real line

We consider a class of sets (Measurable sets) on the real line and the functions (Measurable functions) arising from them.

1.1.Lebesgue Outer Measure:

All the sets considered in this chapter are contained in R, the real line, unless stated otherwise. We will be concerned particularly with intervals I of the form I=[a,b), where a and b are finite, and unless otherwise specified, intervals may be supposed to be of this type. When a=b, I is the empty set ϕ. We will write l(I) for the length I, namely b-a.

Definition 1:

The Lebesgue Outer Measure (or) Outer Measure of a set A is given by $m^*(A) = \inf\sum \ l(I_n)$ where infimum is taken over all finite or countable collection of intervals $[I_H]$ such that $A \subseteq \cup \ I_n$

i.e., $m*(A) = \left\{\inf\sum \ \ell(I_n)/A \subseteq \cup \ I_n\right\}$

Theorem 1:

(i) $m^*(A) \geqslant 0$

(ii) $m^*(\phi) = 0$

(iii) $m^*(A) \leq m^*(B)$ if $A \subseteq B$ (Monotonicity property)

(iv) $m^*([x]) = 0$ for any $x \in R$

Proof:

(i) We know that, $m^*(A) = \{\text{irf}\sum\ell(I_n) \mid A \subseteq \cup \ I_n\}$

Obviously, $\ell(I_n) \geqslant 0 \ \forall n$

$$\inf\sum \ l(I_n) \geqslant 0 \ \forall n$$

$m^*(A) \geqslant 0$

(ii) Clearly, for an empty set

$m^*(\phi) = 0 \quad (\because \text{ length } = 0)$

(iii) We know that

$$m^*(A) = \left\{\inf \sum \ell(I_n) \mid A \subseteq \cup I_H\right\}$$
$$m^*(B) = \left\{\inf \sum \ell(P_n) \mid B \subseteq \cup P_n\right\}$$

since $A \subseteq B \Rightarrow \inf\sum\ell(I_n) \leq \inf\sum\ell(P_n)$

$m^*(A) \leq m^*(B)$.

(iv) Since, $x \in I_n = [x, x + 1/n]$

$$\ell(I_n) = x + 1/n - x$$
$$\therefore \ell(I_n) = 1/n$$
$$\text{Inf} \sum \ell(I_n) = 0$$
$$\therefore M^*([x]) = 0$$

Example 1:

Show that for any set $A, M^*(A) = M^*(A + x)$ where $A + x = [y + x: y \in A]$

(i.e.) the Outer Measure is translation invariant.

Proof:

For each $\varepsilon > 0$, $\exists$ a collection $[I_n]$ such that $A \subseteq \cup I_n$ and

$m^*(A) = \text{iHf} \sum \ell(I_n)$.

$$\Rightarrow m^*(A) \geqslant \sum \ell(I_n) - \varepsilon \;\ldots\ldots(1) \quad \left(\because m^*(A) = \sum \ell(I_n)\right)$$

But clearly, $A + x \subseteq U(I_n + x)$

So for each ε,

$$m^*(A + x) \leqslant \sum \ell(I_n + x) = \sum \ell(I_n) = m^*(A) + \varepsilon$$
$$(By\ \ equation\ (1))$$

(i.e.,) $m^*(A + x) \leq m^*(A) + \varepsilon$
$\Rightarrow m^*(A + x) \leqslant m^*(A) \ldots\ldots\ldots..(2)$

Similarly, we can prove that, $M^*(A - x) \leq M^*(A)$

Replace A by $A + x, M^*(A) \leqslant M^*(A + x)$ ………..(3)

From (2) and (3),

$M^*(A) = M^*(A + x)$

Theorem 2:

The outer measure of an interval equal its length

Proof:

case (i): Suppose that I is a closed interval $I = [a, b]$ (say)

Then for each $\varepsilon > 0$,

$$m^*([a,b]) \leq M^*([a, b+\varepsilon))$$
$$= b - a + \varepsilon$$
$$m^*([a,b]) \leq b - a$$
$$M^*(I) \leq b - a \quad \ldots \ldots (1)$$

$(\because [a,b] \subseteq [a, b+\varepsilon)$ and by theorem 1)

It is enough to prove $m^*(I) \geq b -$ a

For each $\varepsilon > 0$, I may be covered by a collection of intervals $[I_n] = [a_n, b_n)$ s.t $m^*[I] \geqslant \sum l(I_n) - \varepsilon$ ……….(2)

For each n, Let $I_n{}' = \left(a_n - \frac{\varepsilon}{2^n}, b_n\right)$

Then $I_n \subset I_n'$

$$\ell(I_n) = b_n - a_n$$
$$\ell(I_n') = b_n - a_n + \frac{\varepsilon}{2^n}$$
$$\ell(I_n') = \ell(I_n) + \frac{\varepsilon}{2^n}$$
$$\ell(I_n) = \ell(I_n') - \frac{\varepsilon}{2^n} \quad \ldots \ldots \ldots (3)$$

Now, $I \subseteq \cup_{n=1}^{\infty} I_n \subseteq \cup_{n=1}^{\infty} I_n'$

From equation (3),

$$\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \ell(I_n') - \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \qquad \left(\because \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon\right)$$
$$= \sum_{n=1}^{\infty} \ell(I_n') - \varepsilon$$

$\therefore \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \ell(I_n{}') - \varepsilon$ ……… (4)

Theorem 3:

"For a compact set AcR, every open cover has a finite sub cover

There is a finite sub collection of I_n', say $J_1, J_2, \ldots, J_N$ where $J_N = (c_N, d_N)$ covers I

No J_k is obtained in any other we have supposing that $c_1 < c_2 < \cdots < c_N$

$$d_N - c_1 = \sum_{k=1}^{N} (d_k - c_k) - \sum_{k=1}^{N-1} (d_k - c_{k+1})$$

$$< \sum_{k=1}^{N} \ell(J_k) \qquad \dots\dots\dots..(5)$$

From equation (2), (4) and (5)

$$m^*(I) \geqslant \sum \ell(I_n) - \varepsilon$$

$$= \sum_{n=1}^{\infty} \ell(I'_n) - \varepsilon - \varepsilon$$

$$\geqslant \sum_{k=1}^{N} \ell(J_k) - 2\varepsilon$$

$$> d_N - c_1 - 2\varepsilon$$

$$= b - a - 2\varepsilon$$

since $\varepsilon > 0$ is arbitrary, we trave $m^*(I) \geqslant b - a$ ……..(6)

From equation (1) and (6)

$\therefore M^*(I) = b - a$

case (ii)

We have suppose that $I = (a, b]$ where $a > a$

If $a = b, m^*(I) = \ell(I) = 0$

Take $a \neq b, a < b \Rightarrow b - a > 0$

Now we have $0 < \epsilon < b - a$

Consider $I' = [a + \varepsilon, b]$

and Hence $I' \subseteq I$

$m^*(I') \leq m^*(I)$

$m^*(I) \geqslant m^*(I') = \ell(I')$

$= b - a - \varepsilon$

$= \ell(I) - \varepsilon$

$\therefore m^*(I) \geqslant \ell(I) - \varepsilon \qquad \dots\dots\dots\dots(1).$

Consider $I'' = [a, b + \varepsilon)$

$\Rightarrow I \subseteq I''$

$$m^*(I) \leq m^*(I'') = l(I'')$$
$$= b - a + \varepsilon$$
$$= \ell(I) + \varepsilon$$

$\Rightarrow m^*(I) \leq \ell(I) + \varepsilon$ ………. (2)

From equation (1) and (2),

$\ell(I) - \varepsilon \leqslant m^*(I) \leqslant \ell(I) + \varepsilon$

$\Rightarrow m^*(I) = \ell(I) + \varepsilon$

since $\varepsilon > 0$ is arbitrary

$\therefore m^*(I) = \ell(I)$.

case (iii):

Suppose I is an infinite interval. there are 4 types of intervals: $(-\infty, a)$, $(-\infty, a]$, $(a, \infty]$ and $[a, \infty)$

Assume that $I = (-\infty, a]$

For any $M > 0$, there exist k such that

the finite interval $I_M = [k, k + M]$

clearly, $I_M \subseteq I$

$$M^*(I) \geqslant M^*(I_M) = \ell(I_M) = k + M - k$$
$$= M$$
$$\therefore M^*(I) = M$$

But $m^*(I) = \ell(I) = a + \infty = \infty$

$\therefore m^*(I) = \infty = \ell(I)$

the other cases follow similarly.

Hence the outer measure of an interval equal to its length

Theorem 4:

For any sequence of set $\{E_i\}, m^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$

(the property of Countably Sub additivity).

Proof:

For each i, and for any $\varepsilon > 0$

There exists a sequences of Intervals $\{I_{i,j}; j = 1,2, \dots\}$

such that $E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j}$ and ……(*)

$$m^*(E_i) \geqslant \sum_{j=1}^{\infty} \ell(I_{i,j}) - \frac{\varepsilon}{2^i}$$

$$\Rightarrow \sum_{i,j=1}^{\infty} \ell(I_{i,j}) \leq m^*(E_i) + \frac{\varepsilon}{2^i} \qquad \dots\dots\dots (1)$$

From (*), $\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j}$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \qquad \leq m^*\left(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j}\right)$$

$$= \sum_{i,j=1}^{\infty} \ell(I_{i,j})$$

$$= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \ell(I_i, j)\right)$$

$$\leq \sum_{i=1}^{\infty} \left[m^*(E_i) + \frac{\epsilon}{2^i}\right] \qquad \text{(by equation (1))}$$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i}$$

$$= \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon$$

since $\varepsilon > 0$ is arbitrary

$$\therefore m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} m^*(E_i)$$

Example 2:

Show that for any set A and for any $\varepsilon > 0$, there is an open set 0 containing A and such that $m^*(O) \leq m^*(A) + \varepsilon$

consider the sequence of intervals $\{I_n\}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$

$m^*(A) \geqslant \sum_{n=1}^{\infty} \ell(I_H) - \varepsilon/2$ ……… (1)

If $I_n = [a_n, b_n)$ and choose $I'_n = \left(a_n - \frac{\varepsilon}{2^{n+1}}, b_n\right)$

$$\bigcup_{n=1}^{\infty} I_n \subseteq \bigcup_{n=1}^{\infty} I_n'$$

clearly, $A \subseteq \bigcup_{n=1}^{\infty} I_n \subseteq \bigcup_{n=1}^{\infty} I_n'$

Take $0 = \bigcup_{\pi=1}^{\infty} I_\pi'$ is an open set containing A.

Now, $m^*(0) \quad = m^*(\bigcup_{n=1}^{\infty} I_n\,')$

$$\leq \sum_{n=1}^{\infty} m^*(I_n')$$

$= \sum_{n=1}^{\infty} l(I_n')$

$= \sum_{n=1}^{\infty} \left(b_n - a_n + \frac{\varepsilon}{2^{n+1}}\right)$

$= \sum_{n=1}^{\infty} \left(\ell(I_n) + \frac{\varepsilon}{2^{n+1}}\right)$

$= \sum_{n=1}^{\infty} \ell(I_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}}$

$\leq m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= m^*(A) + \varepsilon$

$\therefore m^*(0) \leqslant m^*(A) + \varepsilon.$

1.2.Measurable sets

Definition:

The set E is lebesgue measurable or measurable if for each set A, we have

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

Result

E is measurable if and only if for each set A we have $m^*(A) \geqslant m^*(A \cap E) + m^*(A \cap E^c)$

Proof:

Assume that E is measurable then for any set A,

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

From this, we get

$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$

and $m^*(A) \geqslant m^*(A \cap E) + m^*(A \cap E^c)$

Hence proved.

Conversely, assume that

$m^*(A) \geqslant m^*(A \cap E) + m^*(A \cap E^c)$

It is enough to prove

$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^C)$

We have, $A = (A \cap B) \cup (A \cap B^C)$

$m^*(A) = m^*[(A \cap B) \cup (A \cap B^C)]$

By Countably Sub additive

$m^*(A) \leq m^*(A \cap B) + m^*(A \cap B^C)$

Hence, $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^C)$

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$

$\Rightarrow E$ is measurable.

Example 4:

Show that if $m^*(E) = 0$ then E is measurable.

Solution:

Assume that $m^*(E) = 0$

$$A \cap E \subseteq E$$
$$m^*(A \cap E) \leq m^*(E) = 0 \quad \ldots\ldots..(1)$$
$$A \cap E^c \subseteq A$$
$$m^*(A \cap E^c) \leq m^*(A) \quad \ldots\ldots\ldots.(2)$$

From equation $(1) + (2) \Rightarrow m^*(A \cap E) + m^*(A \cap E^C) \leq 0 + m^*(A)$

$$m^*(A) \geqslant m^*(A \cap E) + m^*(A \cap E^C)$$

$\Rightarrow E$ is measurable (By the previous result).

Definition: ($\sigma -$ algebra)

The class of subsets of an arbitrary space X is said to be a σ-algebra (or) σ-field if x belongs to the class and the class is closed under the formation of countable unions and of complements.

We will denote by M the class of Lebragur measurable sets

Theorems 5:

The class M is a σ-algebra.

Proof:

(i) By definition of Lebesgue measurable sets, we have $k \in M$

$(m^*(A) = \quad m^*(A \cap R) + m^*(A \cap R^C)$
$= m^*(A)$

(ii) For every $E \in M$, to prove $E^c \in M$

For $E \in M$

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$
$= m^*(A \cap (E^c)^c) + m^*(A \cap E^c)$
$m^*(A) = m^*(A \cap E^c) + m^*(A \cap (E^c)^c)$
$\Rightarrow E^c \in M$

(iii) If $\{E_j\}$ is a sequence of sets in M, then prove that $\cup_{j=1}^{\infty} E_j \in M$

Let A be any arbitrary set

if is since $E_1 \in M$

$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C)$
Also $E_2 \in M$
$m^*(A) = m^*(A \cap E_2) + m^*(A \cap E_2^c)$
Take $A = A \cap E_1^c$

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^* \frac{(A \cap {E_1}^c \cap E_2^c)}{A}$$

We have $E_3 \in M$

$m^*(A) = m^*(A \cap E_3) + m^*(A \cap E_3^C)$

Take $A = A \cap E_1^c \cap E_2^c$
$m^*(A \cap E_1^c \cap E_2^c) = m^*(A \cap E_1^c \cap E_2^c \cap E_3) + m^*(A \cap E_1^c \cap E_2^c \cap E_3^c)$

for $n \geqslant 2$

Continuing in this way, we get

$$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left[\left(A \cap E_i \cap \left(\cap_{j<i} E_j^c\right)\right] + m^*\left(A \cap \left(\sum_{j=1}^{n} E_j^c\right)\right]$$

$$m^*(A \cap B)^c = m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left[A \cap E_i \cap \left(\cup_{j<i}^{\infty} E_j\right)^c\right] + m^*\left[A \cap \left(\cup_{j=1}^{n} E_j\right)^c\right]$$

………….. (1)

For any n,

$\bigcup_{j=1}^{n} E_j \subset \bigcup_{j=1}^{\infty} E_j$

$\left(\bigcup_{j=1}^{n} E_j\right)^c \supseteq \left(\bigcup_{j=1}^{\infty} E_j\right)^c$

$A \cap \left(\bigcup_{j=1}^{n} E_j\right)^c \supseteq A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c$

$m^*\left[A \cap \left(\bigcup_{j=1}^{n} E_j\right)^c\right] \geqslant m^*\left[A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right]$

………. (2)

Using equation (2) in (1) we get

$$m^*(A) \geqslant m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i}^{\infty} E_j\right)^c + m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right) \quad \text{... ... (3)}$$

$$\bigcup_{i=1}^{n} E_i \cap \left(\underset{j<i}{U} E_j\right)^c = \bigcup_{i=1}^{n} E_i$$

$$\bigcup_{i=1}^{\infty} E_i \cap \left(\bigcup_{j<i} E_j\right)^c = \bigcup_{i=1}^{\infty} E_i$$

$$A \cap \left[\bigcup_{i=1}^{\infty} E_i \cap \left(\underset{j<i}{U} E_j\right)^c\right] = A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$m^*\left[A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right] = m^* \bigcup_{i=1}^{\infty} \left(A \cap E_i \cap \left(\underset{j<i}{U} E_j\right)^c\right]$$

$$\leq \sum_{i=1}^{\infty} m^*\left(A \cap E_i \cap \left(\underset{j<i}{U} E_j\right)^c\right.$$

$$\textit{From equation } (3) \Rightarrow m^*(A) \geqslant m^*\left[A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right] + + m^*\left[A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right]$$

$$m^*(A) \geqslant m^*\left[A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)\right] + m^*\left[A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

$\therefore \bigcup_{j=1} E_j$ is measurable (by the result)

$\therefore \bigcup_{j=1}^{\infty} E_j \in M$

$\therefore M$ is a σ-algebra

Example 5:

Show that if $F \in M$ and $m^*(F\Delta G) = 0$ then G is measurable.

Solution:

We have $F\Delta G = (F - G) \cup (G - F)$

$m^*(F\Delta G) = 0 \Rightarrow F\Delta G$ is measurable

$\Rightarrow (F - G) \cup (G - F)$ is measurable

$\Rightarrow F - G$ and $G_{-} - F$ are measurable

$F \cap G = F - (F - G)$ is measurable

$\because G = (F \cap G) \cup (G - F)$ is measurable

$\therefore G$ is measurable.

Theorem 6:

If $\{E_i\}$ is any sequence disjoint Hrasurab set. Then $m^*(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i)$

(i.e.) m^* is countably additive on disjoint set of M

Proof:

Given $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint measurable set. $\therefore E_i \cap E_j = \phi, i \neq j$

We know that $\bigcup_{i=1}^{\infty} E_i$ is Measurable

(by definition of M)

Also by theorem 3,

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} m^*(E_i)$$

To prove: $m^*(\bigcup_{i=1}^{\infty} E_i) \geqslant \sum_{i=1}^{\infty} m^*(E_i)$.

$$\bigcup_{i=1}^{\infty} E_i \supseteq \bigcup_{i=1}^{n} E_i$$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geqslant m^*\left(\bigcup_{i=1}^{\pi} E_i\right)$$

$$= \sum_{i=1}^{n} m^*(E_i)$$

As $n \to \infty$, we get

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geqslant \sum_{i=1}^{\infty} m^*(E_i) \quad \dots\dots\dots (2)$$

From equation (1) and (2),

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$$

Note:

- If E is a measurable set, then we will write $m(E)$ in place of $m^*(E)$
- $m(E)$ is called the lebesgue measure of E

Theorem 7:

Every Interval is measurable.

Proof:

We may assume that the interval to be of the form $[a, \infty)$

Fox any set A we wish to show that $m^*(A) \geqslant m^*(A \cap [a, \infty)) + m^*(A \cap (-\infty, a))$

(i.e.) To prove: $m^*(A) \geqslant m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty))$

$let\ A_1 = A \cap (-\infty, a)$

$A_2 = A \cap [a, \infty)$

By definition of m^*, for any $\varepsilon > 0$, there exist a intervals $\{I_n\}$

such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $m^*(A) \geqslant \sum_{n=1}^{\infty} \ell(I_n) - \varepsilon$ …….(1)

Write, $I_n' = I_n \cap (-\infty, a)$

$I_n'' = I_n \cap [a, \infty)$

So that $l(I_n) = \ell(I_n') + \ell(I_n'')$

then, $A_1 \subseteq \bigcup_{n=1}^{\infty} I_n'$

$$\left.\begin{array}{l} A_2 \subseteq \bigcup_{n=1}^{\infty} I_n'' \\ m^*(A_1) \leq \sum_{n=1}^{\infty} R(I_n') \\ m^*(A_2) \leq \sum_{n=1}^{\infty} X(I_n'') \end{array}\right\} (2)$$

Now, $m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} \ell(I_n') + \sum_{n=1}^{\infty} \ell(I_n'')$

$$= \sum_{\Pi=1}^{\infty} \ell(I_{\mathrm{n}})$$

$$\leqslant m^*(A) + \varepsilon \text{ (by (1))}$$

site $\varepsilon > 0$ is arbitrary, we have

$$m^*(A) \geqslant m^*(A_1) + m^*(A_2)$$
$$m^*(A) \geqslant m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty))$$

$\therefore$ $[a, \infty)$ is measurable.

Similarly, we can prove this for other type of intervals.

Hence, every interval is measurable.

Theorem 8:

Let $\mathcal{A}$ be class of subsets of a space X. Then there exists a smallest σ-algebra S containing A. We say that S is the σ-algebra generated by A.

Proof:

Let $[S_\alpha]$ br any collection of σ-algebra. of subset of X. Then, $\bigcap_\alpha S_\alpha$ is a σ-algebra

But, I a σ-algebra containing mark namely the class of all subsets of X.

So, taking the intersection of the, σ-algebra containing $\mathcal{A}$,we get a σ-algebra, necessarily a smallest containing $\mathcal{A}$.

Definition: (Borel sets) - $\mathfrak{B}$

We denote by $\mathfrak{B}$, the σ-algebra generated by the class of intervals of the form $[a, b)$, its member are called the Borel sets of R

Theorem 9:

i) $\mathfrak{B} \subseteq M$, that is every Boral set is treasurable

(ii) $\mathfrak{B}$ is the σ-algebra generated by reach of the following classes: the open intervals, the open sets, the G_δ-sets (countable intersection of open sets, the F_σ - sets (countable union of open sets)

Proof:

(i) M is the class of lebeague measurable sets.

by theorem 4, the class M is a σ-algebra $\mathfrak{B}$ is a σ-algebra generated by the class of intervals of the form $[a, b)$

By theorem 6, Every interval is measurable

$\therefore \mathfrak{B} \subseteq M$

(ii) We first claim that B is the σ - Algebra generated by the class of open intervals

Let B_1 be the σ- Algebra generated by the open intervals.

to prove: $B = B_1$

Every Opens interval is the Union of Sequence of the interval of the form $[a, b)$

it is a Boreal set

$\therefore B_1 \subseteq B$

But every interval (a, b) is the intersection of the sequence of open intervals.

$\therefore B \subseteq B_1$
$\therefore B = B_1$

since, every open set is the union of the sequence of open intervals, the 2nd result follows since, G_S sets and F_σ sets are formed from the open sets using only the countable intersection and complements, and hence the results in these cases follow similarly.

Example 6:

for any set A, there exists a measurable set E containing A and such that $m^*(A) = m(E)$

For any set A and for any $\varepsilon > 0, F$ an open set 0 containing A such that $m^*(0) \leq m^*(A) + \varepsilon$

Take $\varepsilon = 1/n$ and write O_n for the corresponding open set

Then the G_s set, $E = \bigcap_{n=1}^{\infty} O_n$ has the required properties

For every $n, E \subseteq 0_n$

$m^*(E) \leqslant m^*(O_n) \leqslant m^*(A) + \varepsilon$

$\Rightarrow m(E) \leq m^*(A) + 1/n$
$\Rightarrow m(E) \leq m^*(A)$

Now,

$$A \subseteq \bigcap_{n=1}^{\infty} O_n$$

$\Rightarrow A \subseteq E$
$\Rightarrow m^*(A) \leq m^*(E) = m(E)$
$\Rightarrow m^*(A) \leq m(E)$

From (2) and (3)

$m^*(A) = m(E)$

Harte Proved.

lim sup and lim inf of E_i **:**

For any sequence of sets $\{E_i\}$

$$\lim\sup E_i = \bigcap_{n=1}^{\infty} \bigcup_{i\geqslant n} E_i \text{ and}$$

$$\lim\inf E_i = \bigcap_{n=1}^{\infty} \bigcap_{i\geqslant n} E_i$$

Note:

1. $\lim\inf E_i \subseteq \lim\sup E_i$
2. If they are equal, this set is denoted by $\lim E_i$
3. If $E_1 \subseteq E_2 \subseteq \cdots$ then $\lim E_i = \bigcup_{i=1}^{\infty} E_i$
4. If $E_1 \supseteq E_2 \supseteq \cdots$. then $\lim E_i = \bigcap_{n=1}^{\infty} E_i$

Theorem 10:

let $\{E_i\}$ be the sequence of measurable sets Then

(i) If $E_1 \subseteq E_2 \subseteq \cdots$, we have $m(\lim E_i) = \lim m(E_i)$

(ii) If $E_1 \supseteq E_2 \geq \cdots$, and $m(E_i) < \infty$ for each i, then we have $m(\lim E_i) = \lim m(E_i)$.

Proof:

(i) $G_n : E_1 \subseteq E_2 \subseteq \cdots$.

write $F_1 = E_1$ and $F_i = E_i - E_{i-1}$, for $i > 1$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ and the sets F_i are measurable and disjoint.

$$m(\lim E_K) = m\left(\bigcup_{i=1}^{\infty} E_i\right) \text{ (By note 3)}$$

$$= m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} m(F_i) \text{ (By thrm 5)}$$

$$= \lim \sum_{i=1}^{n} m(F_i)$$

$$m(\lim E_\pi) = \lim m\left(\bigcup_{i=1}^{n} F_i\right)$$

$$= \lim m(E_n)$$

$\therefore m(\lim E_i) = \lim m(E_i)$

(ii) $G_m: E_1 \supseteq E_2 \supseteq \cdots.$

$\Rightarrow -E_1 \subseteq -E_2 \subseteq -E_3 \subseteq \cdots$

$E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \cdots$

$\Rightarrow m(\lim(E_1 - E_i)) = \lim m(E_1 - E_i)$

$m(\lim(E_1 - E_i)) = m(E_1) - \lim m(E_i)$

But $\lim\,(E_1 - E_i) = \cup_{i=1}^{\infty}(E_1 - E_i)$ $(By$ note(3)

$\therefore m\lim(E_1 - E_i) = m[U_{i=1}^{\infty}(E_1 - E_i)]$

$$= m\left[E_1 - \bigcap_{i=1}^{\infty} E_i\right]$$

$$= m[E_1 - \lim E_i]$$

$m\lim(E_1 - E_i) = m(E_1) - m\lim E_i$

Equating (1) and (2)

$m(E_1) - \lim m(E_i) = m(E_1) - m\lim E_i$

$\Rightarrow \lim m(E_i) = m\lim E_i$

Example 7:

(i) Show that every non-empty open sets has positive measure.

(ii) The Rational Q are enumerated as $q_1, q_2,$ ...... and the set G is defined by

$G = \cup_{n=1}^{\infty}\left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$ Prove that for any closed set F, $m(G\Delta F) > 0$

Proof:

(i) since non empty open set is the union of disjoints open intervals and Also the outer measure of an interval is its length. Hence, the first (i) follows

(ii) We know that $G\Delta F = (G - F) \cup (F - G)$ $m(G - F) > 0$

$m(G\Delta F) = m(G - F) + m(F - G)$

$m(G - F) > 0$, there's nothing to prove

If $m(G - F) = 0$, then $G - F$ is open

Also, we have $G \subseteq$ F

G contains Q whose closure is R

So $F = R, m(F) = \infty$

$$G = \bigcup_{\pi=1}^{\infty} (q_n - 1/n^2, q_n + 1/n^2)$$

$$m(G) = \sum_{n=1}^{\infty} [q_n + 1/n^2 - q_n + 1/n^2]$$

$$= \sum_{n=1}^{\infty} [2/n^2]$$

$$= 2\sum_{n=1}^{\infty} (1/n^2)$$

$$m(G) = 2(1 + 1/2^2 + 1/3^2 + \cdots) > 0$$

$$\therefore m(F - G) = m(F) - m(G)$$

$$= \infty > 0$$

$$\therefore m(G\Delta F) > 0.$$

Example 8

Show that there exist uncountable sets of zero measure.

Solution:

Here to show that the cantor set P is uncountable and $m(p) = 0$

Construction of cantor set

Consider the interval $[0,1]$

Let $P_0 = [0,1]$, No of intervals- 2^0

First, remove $(1/3,2/3)$.

Let $P_1 = [0,1/3] \cup [2/3,1]$, Then No of intervals- 2^1

Then remove $(1/9,2/9)$ and $(7/9,8/9)$

Let $P_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$, Then No of intervals- 2^2

Continuing in this way, we get

P_k is the union of 2^k closed intervals whose length is $1/3$ K

Then $P = \cap_{n=1}^{\infty} P_n$ is the cantor set

$\because a \in P$ & $1 \in P, P$ is non-empty.

Let $x \in P$

Then $\ni x_n \in P_n -: |x - x_n| < 1/3n$

$\therefore (x_n) \to x$

$\therefore x$ is a limitpoint. in P.

(i.e.,) P has no isolated points.

Also P is a closed set

$\therefore P$ is a perfect set [If p is closed & have no isolated points then p is perfect set]

Hence P is uncountable

[$\because$ A non-empty perfect set is uncountable]

Here P is a countable intersection of closed sets $\therefore P$ is measurable.

Also on each step we remove 2^{k-1} intervals of length $1/3$ K.

$$\begin{aligned}\therefore m(P) &= m([0,1]) - (1/3 + 2/32 + 2^2/33 + \cdots) \\ &= (1-0) - \sum_{k=1}^{\infty} \frac{2^{k-1}}{3k} \\ &= 1 - \frac{1}{3}\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} \\ &= 1 - \frac{1}{3}\frac{1}{1-2/3} \\ &= 1 - \frac{1}{3} \times \frac{3}{1} \\ &= 1 - 1 = 0 \end{aligned}$$

$$\therefore m(P) = 0.$$

Hence the cantor set P is uncountable of of zero measure.

1.3. Regularity:

The next results states that the measurable sets are those which can be approximated closely, in terms of m^*, by open or closed sets.

A non-negative countably additive set function satisfying the conditions (ii) to (iii)* below is said. to be a regular measure.

Theorem 10:

The following statements regarding the set E are equivalent:

(i) E is measurable

(ii) $\forall \varepsilon > 0,70$, an open set, $0 \supseteq E$ such that $m^*(O - E) \leqslant \varepsilon$

(iii) $\exists G, aG_\delta -$ set, $G \supseteq E$ such that $m^*(G - E) = 0,$

(ii)* $\forall \varepsilon > 0,7$ F, a closed set, $F \subseteq E$ such that $m^*(E - E) \leq \varepsilon$

(iii) * $\exists F$, an $F_\sigma -$ set, $F \subseteq E$ such that $m^*(E - F) = 0$

Proof:

(i) ⇒ (ii)

Given E is measurable.

Case 1 m$(E) < \infty$

By Ex:2, for any $\varepsilon > 0,$ their exist an open set $O \supseteq E \ni$:

$m(0) \leqslant m(E) + \varepsilon$ .... (1)

$\therefore E \subseteq 0,\ O = (O - E) \cup E$

Now, $O - E$ & F are disjoint

$\Rightarrow m(O) = m(O - E) + m(E)$ [$\because$ By Additive Property]
$\Rightarrow m(O) - m(E) = m(O - E)$
$\Rightarrow m(O - E) = m(0) - m(E) < \varepsilon$ [by 0]

Case 2: $m(E) = \infty$

$\because \mathbb{R}$ is open, it is the union of countable number of disjoint open intervals

(i.e.,) $\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$

Let $E_n = E \cap I_n$

$\because I_n$ is of finite measure, $m(E_n)$ is finite.

So by case 1, for any $\varepsilon > 0,7$ an open set O_n such that $m(O_n - E_n) \leqslant \frac{\varepsilon}{2^n}$

Take $0 = \bigcup_{n=1}^{\infty} O_n$

⇒ 0 is an open set [$\because$ each on is open]

Also $E_n = E \cap I_n$

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n)$$

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n = E \cap \left(\bigcup_{n=1}^{\infty} I_n\right) = E \cap \mathbb{R} = E$$

$$\Rightarrow E = \bigcup_{n=1}^{\infty} E_n$$

$$\therefore O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (O_n - E_n)$$

$$\Rightarrow m(O - E) \leqslant \sum_{n=1}^{\infty} m(O_n - E_n)$$

$$\leqslant \sum_{n=1}^{\infty} \varepsilon/2n \leqslant \varepsilon$$

(i.e.,), $m(O - E) \leqslant \varepsilon$

(or) equivalently, $m^*(O - E) \leqslant \varepsilon$

To prove: (ii) ⇒ (iii)

By (ii), for each ' n ', let on be an open set such that $E \subseteq O_n$ & $m^*(O_n - E) < Y_n$

Let $G = nO_n$

⇒ G is a G_δ-set [∵ each on is open]

Now, ECO_n for each ' n '

$\Rightarrow E \subseteq\cap O_n$
$\Rightarrow E \subseteq G \;\; \& \;\; G - E \subseteq O_n - E \; (\because G \subseteq O_n)$
$\Rightarrow m^*(G - E) \leq m^*(O_n - E)$
$< V_n$ for 1,2,3, ...

(i.e,), $m^*(G - E) < Y_n$

As $n \rightarrow \infty, m^*(G - E) = 0$

To prove: (iii) ⇒ (i)

By (iii), there exists a Go-set ' G ' containing E such that $E = G - (G - E)$

We know that any G_δ-set is measurable

$\because m(G - E) = 0, G - E$ is measurable

$\because G - E \subset G, E = G - (G - E)$ is measurable

$\therefore E$ is measurable.

To prove: (i) $\Rightarrow (ii)^*$

Suppose E is measurable

$\Rightarrow E^c$ is also measurable

By (ii), $\forall \varepsilon > 0, \exists 0$, an open set, $0 \geq E^c$ such that

$m^*(O - E^c) \leqslant \varepsilon$ ....(1)

We know that $O - E^c = E - O^c$

$\therefore$ (1) $\Rightarrow m^*(E - O^c) \leqslant \varepsilon$

Take $F = O^C$

$\therefore m^*(E - F) \leqslant \varepsilon$

Hence $\forall \varepsilon > 0, \mathcal{F}$, a closed set, $F \subseteq E$ such that

$m^*(E - F) \leqslant \varepsilon$

To prove: (ii)* $\Rightarrow$ (iii)

By (ii)*, for each ' n ', let F_n be a closed set z:

$F_n \subseteq E$ and $m^*_\infty(E - F_n) < 1/n$

Let $F = \bigcup_{n=1}^{\infty} F_n$

$\Rightarrow F$ is an F_σ-set ($\because$ each F_n is closed)

Now, $F_n \subseteq E$ for each ' n '.

$\Rightarrow UF_n \subseteq E$

(i.e.,) $F \subseteq E$

$m^*(E - F) \leqslant m^*(E - F_n) < y_n \ \forall n$

As $n \rightarrow \infty$, $m^*(E - F) = 0$

To prove: (iii) $^* \Rightarrow$ (i)

By (iii) *, $\exists$ an $F_\sigma -$ set ' F ' contained in $E\vartheta$.

$F = E - (E - E) \Rightarrow E = E + (E - E)$

W.K.T any F_0-set is measurable $\therefore m(E - E) = 0, E - F$ is measurable

$\therefore E - F \subseteq E, E = F + (E - F)$ is measurable.

(i.e.,) E is measurable.

Theorem 11:

If $m^*(E) < \infty$, then E is measurable if $\forall \varepsilon > 0, \exists$ disjoint finite intervals $I_1, I_2, \dots, I_n \exists$: $m^*(E \Delta \cup_{i=1}^{n} I_i) < \varepsilon$. We may stipulate that the intervals ${}^{i=1}I_i$ be open, closed or half open.

Proof:

only if Part

suppose that E is measurable.

$\Rightarrow \forall \varepsilon > 0, \exists$ an open set $O \supset E$:

$m(O - E) < \varepsilon$ .... (1)

$\therefore E \subset 0, m(O - E) = m(O) - m(E)$

$\Rightarrow m(O) - m(E) < \varepsilon$ [by (1)]

$\Rightarrow m(0) \leqslant m(E) + \varepsilon$

$\Rightarrow m(0)$ is finite ($\because m(E) < \infty$ (e), $m(E)$ is finite) $\because$ 0 is open, 0 is the union of countable number of disjoint open intervals I_i, $i = 1,2, \dots$

(i.e.,) $0 = \cup_{i=1}^{\infty} I_i$

$\because m(0)$ is finite, $\sum_{i=1}^{\infty} l(I_i)$ is a convergent series.

$\therefore$ Given $\varepsilon > 0$, we can find an ' n ' such that

$\sum_{i=n+1}^{\infty} e(I_i) < \varepsilon$ .... (2)

Using this 'on, we let $U = \hat{U}_{=1}^{n} I_i$

$\Rightarrow U \subset 0$

Now, $E \subset O$ and $U \subset O$

$\therefore E \Delta U = (E - U) \cup (U - E) \subseteq (O - U) \cup (O - E)$ .... (2)

Now,

$$0 - U = \bigcup_{i=1}^{\infty} I_i - \bigcup_{i=1}^{n} I_i = \bigcup_{i=n+1}^{\infty} I_i$$

$$\Rightarrow m(0 - U) = m^*(U_{i=n+1}^{\infty} I_i)$$

$$\leqslant \sum_{i=n+1}^{\infty} m^*(I_i)$$

$$= \sum_{i=n+1}^{\infty} e(I_i)$$

$$< \varepsilon$$

(i.e.,) $m(O-U)<\varepsilon$ .... (4)

Now,

$$(3) \Rightarrow E\Delta U \subseteq (O-U)U(O-E)$$
$$\Rightarrow m(E\Delta U) \leq m(O-U)+m(O-E)$$
$$< \varepsilon+\varepsilon \text{ (by } (1) \& (4)$$
$$= 2\varepsilon = \varepsilon_1$$

$\therefore$ *Their exist* a finite union U of disjoint open intervals such that $m^*(E\Delta U)<\varepsilon$.

If we wish the intervals to be, say, half-open, we first obtain open intervals $I_1, I_2, \ldots, I_n$ as above and then for each ' i ' choose a half-open interval $J_i \subset T_i\theta_i$

$m(I_i-J_i)<\varepsilon/n$

clearly, the intervals J_i are disjoint

We know that for any sets E, F, G we have

$$E \vartriangle F \subseteq (E \vartriangle G) \cup (G \vartriangle F)$$

$$\therefore \left(E\Delta \bigcup_{i=1}^{n} J_i\right) \subseteq \left(E\Delta \bigcup_{i=1}^{n} I_i\right) \cup \left(\bigcup_{i=1}^{n} I_i\Delta \bigcup_{i=1}^{n} J_i\right)$$

$$\Rightarrow E\Delta\hat{U}_{i=1}^{n}J_i \subseteq \left(E\Delta \bigcup_{i=1}^{n} I_i\right) \cup \left(\bigcup_{i=1}^{n} (I_i\Delta J_i)\right)$$

$$\left[\because \hat{U}_{i=1}^{\hat{E}} E_i\Delta\hat{U}_{i=1}^{\hat{U}} F_i = \bigcup_{i=1}^{n} (E_i\Delta F_i)\right]$$

$$\Rightarrow m\left(E\Delta \bigcup_{i=1}^{n} J_i\right) \leq m\left(E\Delta \bigcup_{i=1}^{n} I_i\right) + m\left(\bigcup_{i=1}^{n} (I_i\Delta J_i)\right)$$

$$< \varepsilon + \sum_{i=1}^{n} \varepsilon/n \text{ [by (5)d (6)]}$$
$$= \varepsilon+\varepsilon/n(n)$$
$$= 2\varepsilon$$

$$\text{(i.e.,) } m\left(E\Delta \bigcup_{i=1}^{n} \pi_i\right) < 2\varepsilon = \varepsilon_2$$

$\therefore G$ a finite union $\bigcup_{i=1}^{n} J_i$ of half open intervals such that

$$m\left(E\Delta \bigcup_{i=1}^{n} J_i\right) < \varepsilon.$$

if Part

Assume that for all $\varepsilon > 0,7$ a finite union $U = \bigcup_{i=1}^{n} I_i$ of disjoint open intervals such that $m^*(U\Delta E) < \varepsilon$ ....(7)

To prove: E is measurable.

It is enough to prove that

$\forall\varepsilon > 0, \mathcal{F}$ an openset $0 \supset E\ \ni: m^*(0 - E) \leqslant \varepsilon$

$m^*(O) \leqslant m^*(E) + \varepsilon$ .... (8)

Given $U = \bigcup_{i=1}^{n} I_i$

Define $U' =$ On U

$\Rightarrow u' \subseteq 0$

We know that $O \bigtriangleup E = (O\Delta U') \cup (U'\Delta E)$

$\Rightarrow m^*(O\Delta E) \leqslant m^*(O\Delta U') + m^*(U'\Delta E)$ .... (9)

now,

To prove: $m^*(O\Delta U')$ & $m^*(U'\Delta E)$ seperately.

Now, $U' \subseteq U \Rightarrow U' - E \subseteq U - E$ ....(10)

Also, $E \subseteq 0$

$$\begin{aligned}
\therefore E - U' &= E \cap (\overline{O \cap U})) \ [\because U' = O \cap U] \\
&= E \cap (\bar{O}U\bar{U}) \\
&= (E \cap \bar{O})U(E \cap \bar{U}) \\
&= \phi U(E - U) \ [\because E \subseteq O \Rightarrow E \cap \bar{O} = \phi] \\
&= E - U \quad \ldots\ldots\ldots (11)
\end{aligned}$$

Now,

$$\begin{aligned}
&U' \subseteq U \Rightarrow U' \bigtriangleup E \subset U \bigtriangleup E \\
&\Rightarrow m^*(U'\Delta E) \leqslant m^*(U\Delta E) \\
&< \varepsilon \text{ [by (7)]} \\
&\therefore m^*(U'\Delta E) < \varepsilon \quad \ldots\ldots\ldots (12)
\end{aligned}$$

Now,

$$\begin{aligned}
&E \subseteq 0 \ \& \ U' \subset 0 \\
&\Rightarrow E \subseteq U'U(U'\Delta E) \\
&\Rightarrow m^*(E) \leqslant m^*(U') + m^*(U'\Delta E)
\end{aligned}$$

$\Rightarrow m^*(E) \leqslant m^*(U') + \varepsilon$ .... (13)

$\because u' \leq 0, m^*(O\Delta U') = m^*(0 - U')$

$\Rightarrow m^*(0\Delta U') = m^*(0) - m^*(U')$

$\leqslant m^*(E) + \varepsilon - m^*(U')$ [by (8)]

$\leqslant m^*(U') + \varepsilon + \varepsilon - m^*(U')$ [by (13)]

$= 2\varepsilon$

$\therefore m^*(0\Delta U') \leqslant 2\varepsilon$ - (14)

$\because E \subseteq O, m^*(O \vartriangle E) = m^*(O - E)$

$\therefore$ (9) $\Rightarrow m^*(O - E) \leqslant m^*(0\Delta U') + m^*(U'\Delta E)$

$\leqslant 2\varepsilon + \varepsilon$ (by (12) + (14)

$\Rightarrow m^*(O - E) \leqslant 3\varepsilon$

For all $\varepsilon > 0$, their exist an open set $E \subseteq 0$ $m^*(O - E) \leqslant \varepsilon$

$\therefore$ E is a measurable.

1.4 Measurable Functions:

Sets of infinite measure and functions taking the values ∞ or $-\infty$ occur in a natural way.

To avoid inconvenient restrictions we use the extended real-number system, (i), we add $\infty \& -\infty$ to the real number system with the conventions that

$a + \infty = \infty \quad (a >$ real, or $a = \infty)$

$a \cdot \infty = \infty \quad (a > 0)$

$a \cdot \infty = -\infty \quad (a < 0)$

$\infty \cdot \infty = \infty$

$0 \cdot \infty = 0$

Similarly, for $-\infty$

We do not define $\infty + (-\infty)$

Definition 7:

Let ' f ' be an extended real-valued function defined on a measurable set E. Then ' f ' is a Lebesgue-measurable function (or) a measurable function if, for each $\alpha \in \mathbb{R}$, the set $\{x: f(x) > \alpha\}$ is measurable

In practice the domain of definition of ' f ' will usually be either $\mathbb{R}$ or $\mathbb{R} - F$ where $m(F) = 0$.

Theorem 12:

The following statements are equivalent:

(i) f is a measurable function / $\forall \alpha, [x: f(x) > \alpha]$ is measurable

(ii) $\forall \alpha, [x: f(x) \geqslant \alpha]$ is measurable

(iii) $\forall \alpha, [x: f(x) < \alpha]$ is measurable

(iv) $\forall \alpha, [x: f(x) \leqslant \alpha]$ is measurable

Proof:

To prove: (i) ⇒ (ii)

Suppose, ' f ' is a measurable function

(i.e.), $\forall \alpha, \{x: f(x) > \alpha\}$ is measurable

Now,

$$\{x: f(x) \geqslant \alpha\} \subseteq \{x: f(x) > \alpha - 1/n\} \ \forall n$$

$$\Rightarrow \{x: f(x) \geqslant \alpha\} = \bigcap_{n=1}^{\infty} \{x: f(x) > \alpha - 1/n\}\}$$

We know that, a countable intersection of measurable set is measurable.

$\therefore \prod_{n=1}^{\infty} \{x: f(x) > \alpha - 1/n\}$ is measurable

$\therefore \{x: f(x) \geqslant \alpha\}$ is measurable. Thus (i) ⇒ (ii)

To prove: (ii) ⇒ (iii)

suppose $\forall \alpha, \{x: f(x) \geqslant \alpha\}$ is measurable

$\Rightarrow \{x: f(x) \geqslant \alpha\}^c$ is measurable $\forall \alpha$

(i.e.,), $\{x: f(x) < \alpha\}$ is measurable $\forall \alpha$. Thus (ii) ⇒ (iii)

To prove: (iii) ⇒ (iv)

suppose $\forall \alpha, \{x: f(x) < \alpha\}$ is measurable

For each $n = 1,2, \ldots$

$$\{x: f(x) \leq \alpha\} \subseteq \{x: f(x) < \alpha + 1/n\}$$

$$\Rightarrow \{x: f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x: f(x) < \alpha + 1/n\}$$

Here $\{x: f(x) < \alpha + 1/n\}$ is measurable [By hypothesis] $\Rightarrow \bigcap_{n=1}^{\infty} \{x: f(x) < \alpha + Y_n\}$ is measurable

$\Rightarrow \{x: f(x) \leqslant \alpha\}$ is measurable $\forall \alpha$

Thus (iii) ⇒ (iv)

To prove: (iv) ⇒ (i)

Suppose $\forall \alpha, \{x: f(x) \leqslant \alpha\}$ is measurable

$\Rightarrow \{x: f(x) \leq \alpha\}^c$ is measurable.

(i.e.,) $\{x: f(x) > \alpha\}$ is measurable

(i.e.,) f is a measurable function.

Thus (iv) $\Rightarrow$ (i)

Hence the theorem.

Example 9:

Show that if ' f ' is measurable, then $\{x: f(x) = \alpha\}$ is measurable for each extended real number α .

Solution:

Given f ' is measurable

(i.e.), $\{x: f(x) > \alpha\}$ is measurable $\forall$ real ' α.

Case 1: ' α ' is finite

We know that, $f(x) = \alpha$ if $f(x) \geqslant \alpha \& f(x) \leqslant \alpha$

$\therefore$ $\{x: f(x) = \alpha\} = \{x: f(x) \geqslant \alpha\} \cap \{x: f(x) \leqslant \alpha\}$ is the intersection of two measurable sets.

$\therefore$ $\{x: f(x) = \alpha\}$ is measurable.

case 2: $\alpha = \infty$

$$\{x: f(x) = \alpha\} = \bigcap_{n=1}^{\infty} \{x: f(x) > n\}$$

Now,

$\{x: f(x) > n\}$ is measurable for $n = 1,2, \ldots$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x: f(x) > n\}$ is measurable.

(i.e.,) = ,2$\{x: f(x) = \alpha\}$ is measurable.

case 3: $\alpha = -\infty$

$$\{x: f(x) = \alpha\} = \bigcap_{n=1}^{\infty} \{x: f(x) < -n\}$$

Now, $\{x: f(x) < -n\}$ is measurable for $n = 1,2, \ldots$

$\Rightarrow \prod_{n=1}^{\infty} \{x: f(x) < -n\}$ is measurable

(i.e.,)., $\{x: f(x) = \alpha\}$ is measurable

Hence $\{x: f(x) = d\}$ is measurable for each extended real number ' α '.

Example 10:

The constant functions are measurable.

Solution:

Let ' f ' be a constant function

$\Rightarrow f(x) = c \; \forall x \in \mathbb{R}$

If $\alpha > c$, then $\{x: f(x) > \alpha\} = \phi$

If $\alpha < c$, then. $\{x: f(x) > \alpha\} = \mathbb{R}$

$\because$ both $\phi + \mathbb{R}$ are measurable, $\{x: f(x) > \alpha\}$ is measurable for every ' α '.

(i.e.,) the constant function ' f ' is measurable.

Example 11:

The characteristic function x_A of the set A, is measurable inf A is measurable.

Solution:

Suppose A is measurable

Then the characteristic function χ_A of A is

$$x_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

To prove: x_A is measurable

For $\alpha \in \mathbb{R}$, we have

$$\{x: x_A(x) > \alpha\} = \begin{cases} \mathbb{R} & \text{if } \alpha < 0 \\ A & \text{if } 0 \leqslant \alpha < 1 \\ \phi & \text{if } \alpha \geqslant 1 \end{cases}$$

In each case the set on the right hand side is measurable

$\therefore x_A$ isameasurable function

conversely,

suppose χ_A is a measurable function

To prove: A is measurable.

Take $A = \{x: x_A(x) > 0\}$

Hence A is measurable.

Example 12:

continuous functions are measurable.

Solution:

Let ' f ' be continuous function

Now,

$$\{x: f(x) > \alpha\} = \{x: f(x) \in (\alpha, \infty)\}$$
$$= \{x: x \in f^{-1}(\alpha, \infty)\}$$
$$= f^{-1}(\alpha, \infty)$$

$\therefore \{x: f(x) > \alpha\} = f^{-1}(\alpha, \infty)$

We know that (α, ∞) is open in $\mathbb{R}$

Also ' f ' is continuous

$\therefore$ Inverse image of an openset (α, ∞) is open in 1 ? (6)., $f^{-1}(\alpha, \infty)$ is open in $\mathbb{R}$

(i.e.,)., $\{x: f(x) > \alpha\}$ is open

$\Rightarrow \{x: f(x) > \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

Hence ' f ' is measurable.

Theorem 13:

Let c be any real number and let $f \neq g$ be real-valued measurable functions defined on the some measurable set E. Then $f + c, cf, f + g, f - g$ and fg are also measurable.

Proof:

Let c be any real number

Given $f + g$ are real-valued measurable functions defined on the same measurable set E.

(i) For each ' α ', we have $\{x: f(x) + c > \alpha\} = \{x: f(x) > \alpha - c\}$

$\because$ ' ' is measurable on $E \Rightarrow \{x: f(x) > \alpha - c\}$ is measurable

$\Rightarrow \{x: f(x) + c > \alpha\}$ is measurable $\forall \alpha$

(i.e.,), $f + c$ is measurable.

(ii) If $c = 0$, then cf is itself a constant function $\therefore$ is is measurable (by ex:10)

If $c > 0$, then $\{x: cf(x) > \alpha\} = \{x: f(x) > \alpha/c\}$ is measurable since ' f ' is measurable.

If $c < 0$, then $\{x: cf > \alpha\} = \{x: f(x) < \alpha/c\}$ is measurable since ' f ' is measurable. $\therefore$ af is measurable

(iii) To prove: For every ' α ', $A = \{x: f(x) + g(x) > \alpha\}$ is measurable. Now, $f(x) + g(x) > \alpha \Leftrightarrow f(x) > \alpha - g(x)$

$\Rightarrow f$ a rational number $r_i \theta_0 f(x) > r_i > \alpha - g(x), i = 1,2, \ldots$

$\therefore x \in \{x_i f(x) > r_i\} + x \in \{x_i g(x) > \alpha - r_i\}$ .... (1)

and both the sets are measurable by hypothesis $\Rightarrow x \in \{x: f(x) > \gamma_i\} \cap \{x: g(x) > \alpha - \gamma_i\}$ and this is measurable as it is the intersection of two measurable sets.

Let $B = \bigcup_{i=1}^{\infty} [\{x: f(x) > r_i\} \cap \{x: g(x) > \alpha - r_i\}]$

claim: $A = B$

Let $x \in A$. As $x \in (1), x \in B \therefore A \subseteq B$

Conversely, if $x \in B$ then $x \in A \therefore B \subseteq A$

$\therefore A = B$

Here B is a countable union of measurable sets $\therefore B$ is measurable

$\Rightarrow A$ is measurable.

(i.e.) $f + g$ is measurable.

(iv) By (ii), if $c = -1$, then $-g$ is measurable

$\Rightarrow f - g$ is measurable $[\because f + (-g) = f - g]$

(v) Lemma: ' f ' is measurable $\Rightarrow f^2$ is measurable.

Proof:

If $\alpha < 0$, then $f^2(x) > \alpha \; \forall x \in \mathbb{R}$

$\therefore \{x: f^2(x) > \alpha\} = \mathbb{R}$ and this is measurable.

If $\alpha \geqslant 0$, then $f^2(x) \geqslant \alpha \forall x \in \mathbb{R}$

$$\Rightarrow f(x) \geqslant \pm\sqrt{a}$$
$$\Rightarrow +\sqrt{a} < f(x) < -\sqrt{a}$$

Hence $x \in \{x: f(x) > \sqrt{a}\}$ and $x \in \{x: f(x) < -\sqrt{a}\}$ and both the sets are measurable by hypothesis $\therefore \{x: f^2(x) > \alpha\} = \{x: f(x) > \sqrt{a}\} \cap \{x: f(x) < -\sqrt{a}\}$ is a measurable set.

Hence the lemma.

Now, $f + g$ are measurable

$\Rightarrow f + g, f - g$ are measurable

$\Rightarrow (f + g)^2, (f - g)^2$ are measurable

$\Rightarrow fg = \frac{1}{4}\{(f + g)^2 - (f - g)^2\}$ is measurable.

Hence the theorem.

Corollary:

The results hold for extended real-valued measurable functions except that $f+g$ is not defined whenever $f=\infty$ and $g=-\infty$ or vice versa, and similarly for $f-g$.

For,

$$\{x: f(x)+g(x)>\alpha\}=\bigcup_{i=1}^{\infty}\left(\{x: f(x)>\gamma_i\} \cap\{x: g(x)>\alpha-\gamma_i\}\right) U$$
$$(\{x; f(x)=\infty\}-\{x: g(x)=-\infty\}) \cup(\{x: g(x)=\infty\}-\{x: f(x)=-\infty\})$$

is a measurable set. The case of $f-g$ is similar.

Theorem 14:

Let $\{f_n\}$ be a sequence of measurable functions defined on the same measurable set. Then

(i) $\sup_{1 \leqslant i \leqslant n} f_i$ is measurable for each ' n '.

(ii) $\inf_{1 \leqslant i \leqslant n} f_i$ is measurable for each ' n '.

(iii) sup f_n is measurable

(iv) inf in is measurable

(v) lim sup f_n is measurable

(vi) lime inf in is measurable.

Proof:

Let fin? be a sequence of measurable functions defined on the same measurable set. E '.

(i) Let $\sup\{f_1, f_2, \ldots, f_n\}=f$ on E

claim $\{x: f(x)>\alpha\}=\cup_{1=1}^n\{x: f_l(x)>\alpha\}$

If $f(x)>\alpha$, then $f_i(x)>\alpha$ for some ' i '.

$$\therefore\{x: f(x)>\alpha\} \subseteq \bigcup_{i=1}^n\{x: f_i(x)>\alpha\}$$

If $x \in \cup_{i=1}^n\{x: f_i(x)>\alpha\}$, then $x \in\{x: f_i(x)>\alpha\}$ for some i.

$\Rightarrow x \in\{x: f(x)>\alpha\}$

$\therefore \because \{x: f_i(x)>\alpha\} \leqq\{x: f(x)>\alpha\}$

$$\therefore\{x: f(x)>\alpha\}=\bigcup_{i=1}^n\{x: f_i(x)>\alpha\}$$

Given f_i is measurable for each ' i '

$\Rightarrow\{x: f_i(x)>\alpha\}$ is measurable

$\Rightarrow \cup_{i=1}^{n}\{x: f_i(x) > \alpha\}$ is measurable.

$\therefore \{x: f(x) > \alpha\}$ is measurable $\forall\alpha$

(i.e.,)' f ' is measurable.]

(i) W.K.T $\left\{x: \sup_{1\leqslant i\leqslant n} f_i(x) > \alpha\right\} = \cup_{i=1}^{n}\{x: f_i(x) > \alpha\}$

Given f_i is measurable for each ' i

$\Rightarrow \{x: f_i(x) > \alpha\}$ is measurable

$\Rightarrow \cup_{i=1}^{n}\{x: f_i(x) > \alpha\}$ is measurable

$\therefore \{x: \sup_{1\leqslant i\leqslant n} f_i(x) > \alpha\}$ is measurable (by equation (1))

(ii) To prove $\inf_{1\leq t\leq n} f_i$ is measurable.

We know that $\inf_{i_{1\leqslant i\leqslant n}} f_i = -\sup_{1\leqslant i\leqslant n}(-f_i)$ ……(2)

Given: f_i is measurable for each ' i '.

$\Rightarrow -f_i$ is measurable for each ' i '.

$\Rightarrow \sup_{1\leqslant i\leqslant n}(-f_i)$ is measurable (by(i))

$\Rightarrow -\sup_{1\leqslant i\leqslant n}(-f_i)$ is measurable

i.e., $\inf_{1\leqslant i\leqslant n} f_i$ is measurable (by (2))

(iii) T.P sup f_n is measurable.

We know that $\{x: \sup f_n(x) > \alpha\} = \cup_{n=1}^{\infty}\ \{x: f_n(x) > \alpha\}$ …………(3)

clearly RHS of (3) is measurable.

$\therefore \{x: \sup f_n(x) > \alpha\}$ is measurable.

(v) We know that inf $f_n = -\sup(-f_n)$

$\therefore$ By (iii), inf f_n is measurable.

(v) We know that $\limsup f_n = \inf(\sup_{i\geqslant n} f_i\ )$ …….(4)

By(iii), $\sup_{i\geqslant n} f_i$ is measurable

By (iv) inf $\left(\sup_{i\geqslant n} f_i\right)$ is measurable

$\therefore$ lim sup f_n is measurable (by (4))

(vi)We know that lim inf $f_n = -\lim\sup(-f_n)$ ……..(5)

By (v), $\lim\sup(-f_n)$ is measurable

$\Rightarrow -\lim\sup(-f_n)$ is measurable

(e), lim inf fn_n is measurable (by (5))

Definition 8:

In line with Definition 5 , we say that the function ' f ' is Borel Measurable or a Borel Function if $\forall\alpha, \{x: f(x) > \alpha\}$ is a Borel set.

Note:

Theorems 12,13,14 and their proofs, apply also to Bore functions when 'measurable function' and 'measurable set' are replaced throughout by 'Borel measurable function' and 'Bores set' respectively.

Definition 9:

If a property holds except on a set of measure zero, we say that it holds almost everywhere, usually abbreviated to a.e.

Theorem 15:

Let f be a measurable function and let $f = g$ a.e. Then g is measurable.

Proof:

Let f & g be any two functions.

Given, f is a measurable function

$\Rightarrow \forall$ real $\alpha, \{x: f(x) > \alpha\}$ is measurable ……..(1)

Given, $f = g$ a.e

$\Rightarrow f$ & g have the same domain & $m\{x: f(x) \neq g(x)\} = 0$ ……..(2)

To prove: g is measurable

(i.e.) T.P: $\forall$ real $\alpha, \{x: g(x) > \alpha\}$ is measurable.

Let $E_1 = \{x: f(x) > \alpha\}$

& $E_2 = \{x: g(x) > \alpha\}$

Now, $x \in E_1 \triangle E_2$

$\Rightarrow x \in (E_1 - E_2) \cup (E_2 - E_1)$
$\Rightarrow x \in E_1 - E_2$ (or) $x \in E_2 - E_1$
$\Rightarrow x \in E_1 + x \notin E_2$ (or) $x \in E_2 + x \notin E_1$
$\Rightarrow f(x) > \alpha + g(x) > \alpha$ (or) $g(x) > \alpha d f(x) \times \alpha$
$((0), x \in \{x: f(x) \neq g(x)\}$
$\therefore E_1 \Delta E_2 \subseteq \{x: f(x) \neq g(x)\}$
$\Rightarrow m(E_1 \Delta E_2) \leqslant m(\{x: f(x) \neq g(x)\})$

(i.e.), $m(E_1 \Delta E_2) = 0$ ………(3)

Here E_1 is measurable of $m(E_1 \Delta E_2) = 0$ (by (1) & (3))

By Example:5, E_2 is measurable.

(i.e.), $\{x: g(x) > \alpha\}$ is measurable $\forall \alpha$

(i.e.), g is measurable.

Hence the theorem.

Example 13:

Let $\{f_i\}$ be a sequence of measurable functions converging ace to ' f '. Then ' f ' is measurable.

Solution:

Let $\{f_i\}$ be a sequence of measurable functions $\Rightarrow \lim_{i \to \infty} f_i$ is measurable.

Given, $\{f_i\} \to f$ ae

(i.e.), $\lim_{i \to \infty} f_i = f$ are

From the above theorem, ' f ' is measurable.

Example 14:

If ' f ' is a measurable function, then so are $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$.

Solution:

Given, ' f ' is measurable

$\therefore$ constant function is measurable, ' O ' is measurable By Theorem 14 (i) & (ii),

$\sup\{f, g\}$ & $\inf\{f, g\}$ are measurable

$\therefore$ $\max\{f, g\}$ & $\min\{f, g)$ are measurable.

$\therefore$ $f^+ = \max\{f, 0\}$ of $f^- = -\min\{f, 0\}$ are measurable.

Example 15:

The set of points on which a sequence of measurable functions $\{f_n\}$ converges, is measurable.

Solution:

By Theorem 14, (v) & (vi)),

lim sup f_n & lime inf f_n are measurable

$\Rightarrow$ lim sup f_n $-$ lim inf f_n is measurable

$\therefore$ $\{x$: (lim sup f_n $-$ liminf$f_n)(x) = \alpha\}$ is measurable $\forall \alpha$.

In particular, $\alpha = 0$

$\{x$: (lim sup f_n $-$ limin $f_n)(x) = 0\}$ is measurable (i), $\{x: \text{lim sup } f_n(x) = \text{lim inf } f_n(x)\}$ is measurable

(i), the set of these points for which $\{f_n\}$ converges is measurable.

Definition 10:

Let f be a measurable function. Then inf $\{\alpha: f \leqslant \alpha a \cdot e\}$ is called the essential supremum of f', denoted by ess sup f.

Example 16:

show that $f \leqslant$ es sup f, ale.

Solution:

If ess sup $f = \infty$, then the result is obvious $\{m\{x, f, \% \times \infty\}$ suppose est sup $f = -\infty$.

Then by Definition 10,

$\forall n \in \mathbb{Z}, f \leqslant n \cdot a \cdot e$

$\therefore f = -\infty$, a.e

Suppose that ess sup f is finite

Write $E_n = \{x: f(x) > 1/n + \text{ ess sup } f\}$

$\& \; E = \{x: f(x) > \text{ ess sup } f\}$

$$\therefore E = \bigcup_{n=1}^{0} E_n$$

From Definition, 10,

Clearly, $m(E_n) = 0$

$\therefore m(E) = 0$

$\therefore f \leqslant$ ess sup f, a.e.

Example 17:

Show that for any measurable functions f and g ess sup $(f + g) \leqslant$ ess sup f + ess sup g, and give an example of strict inequality.

Solution:

From example 16,

$f \leqslant$ ess sup f a.e $\quad g \leqslant$ ess sup g a.e

$\Rightarrow \quad f + g \leqslant$ ess sup f + ess sup g a.e

$\Rightarrow \quad$ ess sup $(f + g) \leqslant$ ess sup f + ess sup g

Example of strict inequality

Let $f = x_{[-1,0)} - x_{[0,1]}$ and $g = -f$

Then $f + g = 0$
& ess sup $f = 1$ & ess sup $g = 1$
$\therefore$ ess sup $f +$ ess sup $g = 1 + 1 = 2$
$0 < 2$

Definition 11:

Let f be a measurable function; Then sup $\{\alpha : f \geqslant \alpha \; a.e\}$ is called the essential infimum of' f denoted by ess inf f.

Example 18:

Ess sup f $= -$ ess inf$(-f)$

Solution:

$$\begin{aligned} \text{ess sup} f &= \inf\{\alpha : f \leqslant \alpha \; a \cdot e\} \\ &= \inf\{\alpha : -f \geqslant -\alpha \; a \cdot e\} \\ &= -\sup\{-\alpha : -f \geqslant -\alpha \; a \cdot e\} \\ &= -\text{ess in} f(-f) \end{aligned}$$

$\therefore$ ess sup f $= -$ess in$f(-f)$.

Note:

So results analogous to those holding for ess sup f, for example those of Examples 16&17, hold also for iss inf f, with the obvious alterations. Definition 12

If f is a measurable function and iss sup $|f| < \infty$, then f is said to be essentially bounded. If

Example 19:

Let f be a measurable function and B a Bores set; then $f^{-1}(B)$ is a measurable set.

Solution:

We have $f^{-1}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} f^{-1}(A_i)$

& $f^{-1}(A^C) = (f^{-1}(A))^c$

The class of sets whose inverse image under f are measurable forms a σ-algebra

But this class contains the intervals. $\therefore$ it must contain all Borel sets.

2.5 Borel and Lebesgue Measurability:

Note:

- B- Borel set
- M -class of Lebesgue -measurable set
- $P(iR)$ - class of all subsets of $\mathbb{R}$

- (Theorem 8): $B \subseteq M \subseteq P(\mathbb{R})$
- (Ex:11): x_A is measurable $\Leftrightarrow A$ is measurable
- x_A is Borel measurable $\Leftrightarrow A$ is a Borel set

Theorem 16:

Let E be a measurable set. Then for each ' y ' the set $E+y=\{x+y: x \in E\}$ is measurable and the measures are the same.

Proof:

Given E is measurable

By theorem 10, $\forall \varepsilon>0, \exists$ an open set $0 \supseteq E \ni: m(O-E) \leqslant \varepsilon$

Now, 0 is open $\Rightarrow 0+y$ is also open

We have $E \subseteq 0 \Rightarrow E+y \subseteq 0+y$

Now,

$$(O+y)-(E+y)=(O-E)+y$$
$$\Rightarrow m[(O+y)-(E+y)]=m[(O-E)+y]$$
$$\equiv m(O-E) \qquad \text{[by ex: 1 } m^*(A) m^*(A+x)]$$
$$\leqslant \varepsilon \text{ (by (1))}$$

(i.e.), $m[(0+y)-(E+y)] \leqslant \varepsilon$

Hence $\forall \varepsilon>0$, their exist an open set $0+y \geq E+y$ such that $m[(0+u)-(E+y)] \leqslant \varepsilon$.

Thus $E+y$ is measurable

Again example 1, we have

$m^*(E)=m^*(E+y)$

$\because E$ & E+y are measurable,

$m(E)=m(E+y)$

Hence the theorem.

Theorem 17:

There exists a non-measurable set.

Proof:

Let $x, y \in[0,1]$

Let $x \sim y$ if $y-x \in Q_1=Q \cap[-1,1]$

(1) claim: ' $\sim$ ' is an equivalence relation on $[0,1]$

(i) Reflexive: $x \in [0,1]$. Then $x - x = 0 = \frac{0}{1} \in Q_1 \quad \therefore x \sim x$.

(ii) Symmetric: $x, y \in [0,1]$. Suppose $x \sim y \Rightarrow y - x \in Q_1$

$\therefore x - y = -(y - x) \in Q_1 \therefore x - y \in Q_1$
$\therefore y \sim x$. Hence $x \sim y \Rightarrow y \sim x$.

(iii) Transitive: $x, y, z \in [0,1]$. Suppose $x \sim y$ of $y \sim z$

$\Rightarrow x - y \in Q_1$ & $y - z \in Q_1$
$\Rightarrow (x - y) + (y - z) \in Q_1$
$\Rightarrow x - z \in Q_1$
$\Rightarrow x \sim z$
$\therefore x \sim y + y \sim z \Rightarrow x \sim z$.

$\therefore$ ' $\sim$ ' is an equivalence relation on $[0,1]$

$\therefore x \sim y \Leftrightarrow [0,1] = UE_\alpha, E_\alpha \rightarrow$ disjoint sets

where $x + y$ are in same E_α.

$\because Q_1$ is countable, Each E_α is countable.

$\because [0,1]$ is uncountable, there are uncountable many set E_α.

By the Axiom of Choice,

we consider a set V in $[0,1]$ containing just one element x_α from each E_α.

To prove: V is not a measurable set.

suppose V is measurable

let $\{r_i\}$ be an enumeration of Q_1

For each n, write $V_n = V + r_n$

claim: (i) $V_n \cap V_m = \phi, n \neq m$

(ii) $UV_n = [0,1]$

(i) $V_n \cap V_m = \phi$

suppose $V_n \cap V_m \neq \phi$

Let $y \in V_n \cap Vm$

$\Rightarrow y \in V_n$ and $y \in V_m$
$\Rightarrow yx_\alpha + x_\beta \in V \exists$:
$y = x_\alpha + r_n \& y = x_\beta + r_m$
$\Rightarrow x_\alpha + r_n = x_\beta + r_m$
$\Rightarrow x_\beta - x_\alpha = r_n - r_m \in Q_1$

(i.e.). $x_\beta - x_\alpha \in Q_1$

$\therefore x_\alpha \sim x_\beta$

$\therefore x_\alpha$ & x_β are in the same class E_α

$\Rightarrow\Leftrightarrow (\because x_\alpha, x_\beta \in V)$

$\therefore$ Our assumption is wrong.

$\therefore\ V_n \cap V_m = \phi$ for $n \neq m$.

(ii) $UV_n = [0,1]$

Now, let $x \in [0,1]$

$\Rightarrow x \in E_\alpha$ for some α

$\Rightarrow\ x = x_\alpha + r_n$

$\Rightarrow x \in V_n$
$\Rightarrow x \leqslant UV_n$
$\therefore [0,1] \leqslant UV_n$

Now, Let $x \in UV_n$

$\Rightarrow x \in V_n$ for some n
$\Rightarrow x \in V$
$\Rightarrow x \in [0,1]$
$\therefore UV_n \subseteq [0,1]\ \ldots\ldots..(1)$

Now, By our assumption, V is measurable

By theorem 16, we have V_n isolso measurable & $m(V) = m(v_n)$

Now,
$(1)\ \Rightarrow [0,1] \equiv UV_n$
$\Rightarrow m([0,1) = m(UV_n)$
$\Rightarrow\ 1 = \sum\ m(V_n)$
$\Rightarrow\ 1 = \sum\ m(v)$

Here the sum $\Sigma m(v) = 0$ (or) ∞

$\therefore\ 1 \neq 0$ & $1 \neq \infty$

$\therefore$ our assumption is wrong

$\therefore V$ is not a measurable set.

Theorem 18:

Not every measurable set is a Borel set.

Proof:

Let $x \in [0,1]$

Write x in binary form as

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$$

with $\varepsilon_n = 0$ (or) 1 , choosing a non-terminating expansion for each $x > 0$.

Define a cantor function $f: [0,1] \to P$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}$$

The value of ' f ' lie entirely in the cantor set p. Here ε_n is a measurable function of x

$\therefore f$ is a measurable function. ……… (1)

since the value $f(x)$ defines the sequence $\{\varepsilon_n\}$ in the expansion $\sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}$ uniquely, so ' x ' is determined uniquely.

$\therefore f$ is a one-to-one mapping from $[0,1]$ onto its range.

(i.e.)., f is a bijective function …….(2)

let B & M be the class of Borel set & Lebesgue measurable set.

We know that, $B \subseteq TM$. "TIP. $\beta \neq TM$ /. suppose $\beta = \pi$

suppose $\mathcal{B} =$ M

Let B be a Borel set. …….. (3)

By example:19, $f^{-1}(B)$ is a measurable set [by equation (1)&(3)]

Let V be a non-measurable set in $[0,1]$

Then $B = f(v) \subset P$

$\Rightarrow m(B) \leqslant m(P) = 0$

$\Rightarrow m(B) = 0$

$\Rightarrow B$ is measurable.

Now, $B \equiv f(V)$

$\Rightarrow f^{-1}(B) = V$ ($\because f$ is $1-1$)

$\Rightarrow f^{-1}(B)$ is non-measurable

$\Rightarrow\Leftarrow$

$\therefore \beta \subset M$

Hence Not every measurable set is a Borel set.

Example 20:

Let T be a measurable set of positive measure and let $T^* = \{x - y : x \in T, y \in T\}$. show that T^* contains an interval $(-\alpha, \alpha)$ for some $\alpha > 0$.

Solution:

Let T be a measurable set of positive measure Let $T^* = \{x - y / x \in T, y \in T\}$.

By Theorem 10,

T contains a closed set c of positive measure.

Now, $m(c) = \lim_{n\to\infty} m(c \cap [-n, n])$

$\therefore$ we may assume that C is bounded

By Theorem 10,

$\exists$ an open set $U, U \supset C \; such \; that \; m(U - C) < m(C)$

Define the distance between two sets A and B to be $d(A, B) = \inf\{|x - y| : x \in A, y \in B\}$

clearly, $|x - y|$ is a continuous function of $x \neq y$. If $A \& B$ are disjoint closed sets one of which is bounded, the distance between $A + B$ is positive. Let $\alpha = d(c, U^c)$

$\therefore \alpha > 0$

Let x be any point of $(-\alpha, \alpha)$ (i.e.,)$x \in (-\alpha, \alpha)$

To show that $c \cap (c - x) \neq \phi$

Now, $c - x = \{y : y + x \in c\}$

$\forall x \in (-\alpha, \alpha), \exists z \in C \rightarrow: z' = z + x \in C$

$\Rightarrow x = z' - z \in T^*$ (i.e.),$x \in T$

$\because |x| < \alpha$

$\Rightarrow c - x \subset u$ [by the definition of α]

$$\begin{aligned}\therefore m(c - (c - x)) &\leqslant m(u - (c - x)) \\ &= m(u) - m(c - x) \\ &= m(u) - m(c)(by \text{ Theorem } 16) \\ &< m(c)\end{aligned}$$

$\therefore m(cn(c - x)) > 0$

$\therefore cn(c - x) \neq \phi$

Example 21:

Suppose that f is any extended real-valued function which for every x and y satisfies

$f(x) + f(y) = f(x + y)$

(i) Show that f is either everywhere finite or everywhere infinite.

(ii) Show that if f is measurable and finite, then $f(x) = x.f(1)$ for each x.

Solution:

Let f be any extended real-valued function & $f(x) + f(y) = f(x + y)$ $\forall x$ & y. …….(1)

(i) f cannot take both values ∞ & $-\infty$

suppose $f(x) = \infty$ for some x.

Then $f(x + y) = f(x) + f(y) = \infty + f(y) = \infty$ &

$\therefore f(x + y) = \infty\ \forall y.$

$\therefore f = \infty$ everywhere

similarly, if $f(x) = -\infty$ for some x

Then $f(x + y) = f(x) + f(y) = -\infty + f(y) = -\infty$

$\therefore f(x + y) = -\infty\ \forall y$

$\therefore f = -\infty$ everywhere.

(ii) (1) gives $f(nx) = n \cdot f(x)\ \forall x$ & $\forall n > 0$ [By induction]

$\Rightarrow f(x/n) = n^{-1} f(x).$

$\Rightarrow f\left(\frac{mx}{n}\right) = mn^{-1} f(x)$

In particular, $f(r) = r \cdot f(1)\ \forall r \in Q.$

$\because f$ is finite, $\exists$ a measurable set E :

$m(E) > 0$ & $|f| < M$ on E.

Let $E^* = \{x - y : x \in t, y \in T\}$

Let $z \in E^* \Rightarrow z = x - y$ where $x_2 y \in E$

Then $|f(z)| = |f(x - y)| = |f(x) - f(y)| \leqslant M + M = 2M$

(i.e.),$|f(z)| \leqslant 2M$

By Ex :20, E^* contains an interval $(-\alpha, \alpha)$ with $\alpha > 0$

(i.e.),$(-\alpha, \alpha) \subseteq E^*$ Now, if $|x| < \alpha/n$

$|f(nx)| \leq 2M$

$\Rightarrow |f(x)| \leq \frac{2M}{n}$ for each n

Let x be real of let r be a rational →: $| r - x |< \alpha/n$

Now,
$$\begin{aligned} |f(x) - xf(1)| &= |f(x) - f(\gamma) + (\gamma - x)f(1)| \\ &= |f(x-\gamma) + (\gamma - x) \cdot f(1)| \\ &\leqslant \tfrac{2M}{n} + \tfrac{\alpha}{n}|f(1)| \qquad \forall n \end{aligned}$$

As $n \to \infty,\ f(x) \equiv x \cdot f(1)$

Unit II

Integration of Functions of a Real variable - Integration of Non- negative functions - The General Integral - Riemann and Lebesgue Integrals.

Chapter - 2 Sec 2.1-2.3

Integration of Functions of a Real variable

In analysis it is often convenient to replace an expression of the form $\int \sum f_n dx$ by $\sum \int f_n dx$ (or) $\int \lim f_n dx$ by $\lim \int f_n dx$ (or) $\int \lim_{\alpha \to \alpha_0} f_\alpha dx$ by $\lim_{\alpha \to \alpha_0} \int f_\alpha dx$

In this chapter we give a definition of an integral which applies to a large class of Lebesgue measurable functions and which allows the interchange of integral and sum or limit in very general circumstances.

2.1. Integration of Non-negative functions:

We consider first the class of non-negative measurable functions, define the integral of such a function and examine the properties of the integral. For the present we will suppose these functions to be defined for all real ' x '.

Definition:

A non-negative finite-valued function $\phi(x)$, toking only a finite number of different values, is called a simple function.

If $a_1, a_2, \ldots, a_n$ are the distinct values taken by Q and $A_i = \{x: \varphi(x) = a_i\}$, then clearly

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$$

where χ_{A_i} is called the characteristic function of A_i.

(i.e.,) $x_{A_i}(x) = \begin{cases} 0 & x \in A_i \\ 1 & x \notin Ai \end{cases}$

Result:

φ is measurable $\Leftrightarrow$ The sets Ai are measurable

Proof:

Assume that φ is measurable.

$$A_i = \{x/\varphi(x) = a_i\} \forall i$$
$$A_i = \varphi^{-1}(\{a_i\}) \ \ldots \ldots \ldots . (1)$$

$\because$ φ is measurable, ϕ^{-1} is also measurable.

$\therefore$ A_i is measurable. (by (1))

Conversely,

suppose that A_i is measurable

By example $11, x_{A_i}$ is measurable

$\therefore$ $\sum_{i=1}^{n} a_i x_{A_i}(x)$ is measurable.

$\therefore$ Q is measurable. If

Definition:1

Let ϕ be a measurable simple function. Then

where, $\int \dot{\varphi} dx = \sum_{i=1}^{n} a_i m(A_i)$

$a_1, a_2, \dots, a_n$ are distinct values taken by φ and

$A_i = \{x: \varphi(x) = a_i\}$ is called the integral of ϕ.

Example 1:

let the sets A_i be defined as above. Then $A_i \cap A_j = \phi,\ i \neq j$ and $\bigcup_{i=1}^{n} A_i = \mathbb{R}$.

Definition 2:

For any non-negative measurable function ' f ', the integral of ' f ' is given by $\int fdx = \sup \int \phi dx$ where the supremum is taken over all measurable simple functions $\varphi,\ \varphi \leqslant f$.

Definition 3:

For any measurable set E, and any non-negative measurable function ' f ', $\int_E fdx = \int fx_E dx$ is the integral of ' f ' over E. If the set $E = [a, b]$, then $\int_E fdx = \int_a^b fdx$.

If $a > b$, then $\int_a^b fdx = -\int_b^a fdx$. This integral $\int_a^b fdx$ is referred to as lebesgue Integral

Example 2:

If ϕ is a measurable simple function, Definition I and definition 2 both give a value for its integral. show that these values are the same.

Solution:

Let Q be a measurable simple function.

Write $\int^{*} \varphi dx = \sup \int \psi dx$ ....(1)

where ψ is any measurable simple function $- - \psi \leqslant \Phi$

write $\int \phi dx = \sum_{i=1}^{n} a_i m(A_i)$ ……….(2)

where $a_1, a_2, \dots, a_n$ are distinct values taken by φ and

$A_i = \{x: \varphi(x) = a_i\}$

To prove: $\int \varphi dx = \int \varphi^* \, dx$.

clearly, $\int \varphi dx \leqslant \int \varphi^* \, dx$ ……… (3)

If $\psi \leqslant \phi$ is a measurable simple function with

distinct values bj $(j = 1,2, \dots, m)$ and $\psi = \sum_{j=1}^{m} b_j x_{B_j}$,

then $$\int \psi dx = \sum_{j=1}^{m} b_j m(B_j) = \sum_{j=1}^{m} \sum_{i=1}^{n} b_j m(B_j \cap A_i)$$

where $b_j \leqslant a_i$ if $m(B_j \cap A_i) > 0$

$$\therefore \int \psi dx = \sum_{j=1}^{m} \sum_{i=1}^{n} b_j m(B_j \cap A_i)$$

$$\leqslant \sum_{j=1}^{m} \sum_{i=1}^{n} a_i m(B_j \cap A_i) \qquad [\because b_j \leqslant a_i]$$

$$= \sum_{i=1}^{n} a_i \cdot m(A_i)$$

$$= \int \phi dx$$

$$\therefore \int \psi dx \leqslant \int \phi dx$$

$$\Rightarrow \sup \int \psi dx \leqslant \int \phi dx$$

(i.e.,) $\int \varphi^* \, dx \leqslant \int \varphi dx$

From (3) & (4) we get

$$\int \phi dx = \int \varphi^* \, dx \, .$$

Theorem 1:

If ϕ is a measurable simple function and $\varphi(x) = \sum_{i=1}^{n} a_i x_{A_i}(x)$, where $a_1, a_2, \dots, a_n$ are the distinct values taken by ϕ and $A_i = \{x: \varphi(x) = a_i\}$, then (i) $\int_E \phi dx = \sum_{i=1}^{n} a_i m(A_i \cap E)$ for any measurable set E, (ii) $\int_{A \cup B} \varphi dx = \int_A \varphi dx + \int_B \varphi dx$ for any disjoint measurable

(iii) $\int a\varphi dx = a\int \varphi dx$ if $a > 0$.

proof:

Let Q be a measurable simple function

Let $Q(x) = \sum_{i=1}^{n} a_i x_{A_i}(x)$

where $a_1, a_2, \dots, a_n$ are the distinct values taken by φ and $A_i = \{x: \phi(x) = a_i\}$

Let E be any measurable set.

(i) $To\ prove$: $\int_E Qdx = \sum_{i=1}^{n} a_i m(A_i \cap E)$

Now, $\int_E Qdx = \int \phi x_E dx$ [by definition 3]

$\Rightarrow$ $\int_E \phi dx = \sum_{i=1}^{n} a_i \cdot m(Ai \cap E)$ [by definition 1]

iii) Let A & B be any disjoint measurable sets

To prove: $\int_{A\cup B} \phi dx = \int_A \phi dx + \int_B \phi dx.$

Now, $\int_A \varphi dx + \int_B \varphi dx = \sum_{i=1}^{n} a_i m(A_i \cap A) + \sum_{i=1}^{n} a_i m(A_i \cap B)$

$$= \sum_{i=1}^{n} a_i m(A_i \cap (A \cup B))$$

$$= \int_{A\cup B} Qdx$$

$$\therefore \int_{A\cup B} Qdx = \int_A \varphi dx + \int_B \varphi dx.$$

(iii) Let $a > 0$

To prove: $\int a\varphi dx = a\int \varphi dx.$

As φ takes the value a_i, aφ takes the distinct values aa_i.

$$\therefore \int a\varphi dx = \sum_{i=1}^{n} aa_i \cdot m(A_i)$$

$$= a\sum_{i=1}^{n} a_i m(A_i)$$

$$= a\int \phi dx.$$

$$\therefore \int a\varphi dx = a\int \phi dx.$$

Example 3:

Show that if f is a non-negative measurable function, then $f = 0$ a.e. $\Leftrightarrow \int f dx = 0$.

Solution:

Let ' f ' be a non-negative measurable function Let ' ϕ ' be a measurable simple function

$\ni: \varphi \leqslant f$ ……… (1) Suppose $f = 0$ a.e

clearly, $\int \varphi dx = 0$ by (1)

$\therefore \int f dx = \sup \int \phi dx = 0$

$\therefore \int f dx = 0$

conversely,

Suppose that $\int f dx = 0$

Let $E_n = \{x: f(x) \geqslant 1/n\}$

Then

$$\int fdx \geqslant \int \frac{1}{n} x_{E_n} dx$$

$$= n^{-1} \cdot m(E_n)$$

$$\therefore \int fdx \geqslant n^{-1} m(E_n)$$

$$\Rightarrow 0 \geqslant n^{-1} m(E_n) \; [\because \int fdx = 0]$$

$$\Rightarrow m(E_n) = 0$$

Now, $\{x: f(x) > 0\} = \cup_{n=1}^{\infty} E_n$

$\therefore f = 0$ a.e.

Theorem 2:

Let f of g be non-negative measurable functions

(i) If $f \leqslant g$, then $\int f dx \leqslant \int g dx$

(ii) If A is a measurable set and $f \leqslant g$ on A, then $\int_A f dx \leqslant \int_A g dx$

(iii) If $a \geqslant 0$, then $\int a f dx = a \int f dx$

(iv) If A & B are measurable sets and $A \subset B$, then $\int_A f dx \geqslant \int_B f dx$.

Proof:

Let f & g be non-negative measurable functions

(i) Given: $f \leqslant g$

To prove: $\int f dx \leq \int g dx$

Now,

$$\int fdx = \sup\left\{\int \phi dx: \phi \leqslant f\right\} \text{ [}\because By \text{ def n:2]}$$

$$\leqslant \sup\left\{\int \phi dx: \varphi \leqslant g\right\} (\because f \leqslant g\}$$

$$= \int gdx.$$

$$\therefore \int fdx \leqslant \int gdx.$$

(ii) Given $f \leqslant g$

Let A be a measurable function.

To prove: $\int_A fdx \leqslant \int_A gdx$.

Now,

$$\int_A fdx = \int_f fx_A dx \text{ [By def: 3]}$$

$$\leqslant \int_A gx_A dx \text{ [By (i)]}$$

$$= \int_A gdx.$$

$$\therefore \int_A fdx \leqslant \int_A gdx.$$

(iii) Given: $a \geqslant 0$

T.P: $\int afdx = a\int fdx$.

If $a = 0$, then obviously, $\int afdx = a\int fdx$.

suppose $a > 0$. Then φ is a measurable simple function with $\varphi \leqslant af$ ifs $\phi = a\psi$, where ψ is a simple function $t: \psi \leqslant f$

$$\therefore \int \varphi dx = \int a\psi dx$$

$$\Rightarrow \int \varphi dx = a \int \psi dx \text{ [By theorem 1:(iii)]}$$

$$\therefore \int afdx = \sup \int \varphi dx$$

$$= a \cdot \sup \int \psi dx$$

$$= a \cdot \int fdx$$

$$\therefore \int afdx = a \cdot \int fdx$$

(iv) Let A & B are measurable sets & $A \supseteq B$ I. $\int_A fdx \geqslant \int_B fdx$

We know that $fx_A \geqslant fx_B$ ($\because A \perp B$)

$$\Rightarrow \int_A fx_A dx \geqslant \int fx_B dx \; (By(i))$$

$$\Rightarrow \int_A fdx \geqslant \int_B fdx$$

Theorem 3 [Fatou's Lemma]:

Let $\{f_n, n = 1,2, \dots\}$ be a sequence of non-negative measurable functions.

Then $\lim\inf \int f_n dx \geqslant \int \lim\inf f_n dx$.

Proof:

Let $\{f_n, n = 1,2, \dots\}$ be a sequence of non-negative measurable functions.

To prove: $\int \lim\inf f_n dx \leqslant \lim\inf \int f_n dx$.

Let $f = \lim\inf f_n$

Then f is a non-negative measurable function.

$\therefore$ To prove: $\int fdx \leqslant \text{liminf} \int f_n dx$.

(i.e.,) To prove: For every measurable simple function $Q \leqslant f$, $\int \phi dx \leqslant \text{limln} f \int fndx$.

Case(i) $\int \varphi dx = \infty$

Then for some measurable set A, we have

$m(A) = \infty$ and $\varphi > a > 0$

(i.e.,) $A = \{x: \phi(x) > a\}$

Define $g_k(x) = \inf_{j \geqslant k} f_j(x)$ of the measurable set

$A_n = \{x: g_k(x) > a\}$ for $k \geqslant n$

Let $x \in A_n \Rightarrow g_k(x) > a \ \forall k \geqslant n$

$\Rightarrow g_k(x) > a \ \forall k \geqslant n+1$

$\Rightarrow x \in A_{n+1}$

$A_n \subseteq A_{n+1}$

Also $g_k(x) = \inf_{j \geqslant k} f_j(x)$

$\leqslant \inf_{j \geqslant k+1} f_j(x)$

$= g_{k+1}(x)$

$\therefore g_k(x) \leq g_{k+1}(x)$

$\therefore g_k(x)$ is monotone increasing

Now, $\lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} \inf_{j \geqslant k} f_j(x) = f(x) \geqslant \varphi(x)$

$\therefore \lim_{k\to\infty} g_k(x) \geqslant \varphi(x)$

Let $x \in A \Rightarrow \varphi(x) > a$

$\Rightarrow \lim_{k\to\infty} g_k(x) \geqslant \phi(x) > a$

$\Rightarrow \lim_{k\to\infty} g_k(x) > a$

$$\Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$$

$$\therefore A \subseteq \bigcup_{n=1}^{\infty} A_n$$

Taking measure, $m(A) \leqslant m(\cup_{n=1}^{\infty} A_n)$

$\Rightarrow m(A) \leqslant m\left(\lim_{n\to\infty} A_n\right)$

$\Rightarrow m(A) \leqslant \lim_{n\to\infty} m(A_n)$

$\because m(A) = \infty, \lim_{n\to\infty} m(A_n) \geqslant \infty$

$\Rightarrow m(A_n) \geqslant \infty$

Now, $\because g_n(x) = \text{Inf}_{k \geqslant \text{n}} f_k(x)$

$\leqslant f_n(x) \qquad \forall n$

$$f_n \geqslant g_n \quad \forall n$$
$$\Rightarrow \int f_n dx \geqslant \int g_n dx$$
$$> \int_{A_n} g_n dx$$
$$> a \int_{A_n} dx$$
$$= a \int x_{A_n} dx$$
$$= am(A_n)$$
$$\geqslant \infty$$
$$\therefore \int f_n dx \geqslant \infty$$
$$\Rightarrow \lim\inf \int f_n dx \geqslant \infty = \int \varphi(x)$$

$$\int \varphi(x) \leq \lim\inf \int f_n dx$$

case (ii) $\int \varphi dx < \infty$

Define $B = \{x: \phi(x) > 0\} \therefore m(B) < \infty$

Let M be the largest value of $\phi. (i.e.,) \varphi \leq M$

Let $0 < \varepsilon < 1$. Define $B_n = \{x: g_k(x) > (1-\varepsilon)\varphi(x)\}, \forall k \geqslant n$

$\Rightarrow B_n$ are measurable.

If $x \in B_n$, then $g_k(x) > (1-\varepsilon)\varphi(x) \forall k \geqslant n$
$$\Rightarrow g_k(x) > (1-\varepsilon)\varphi(x) \forall k \geqslant n+1$$
$$\Rightarrow x \in B_{n+1}$$
$$\therefore B_n \subseteq B_{n+1} \ \forall n$$

Also $B = \bigcup_{n=1}^{\infty} B_n \cdots$

$$\Rightarrow B - \bigcup_{n=1}^{\infty} B_n \equiv \varphi.$$

$$\Rightarrow \bigcap_{n=1}^{\infty} (B - B_n) = \varphi$$

$B - B_n \geqslant B - B_{n+1}$ ($B_n \subseteq B_{n+1}$)

$\therefore \{B - B_n\}$ is a monotone decreasing sequence and $\bigcap_{n=1}^{\infty} (B - B_n) = \varphi$

As $m(B) < \infty, \exists N \rightarrow: m(B - Bn) < \varepsilon \; \forall n \geqslant N$

Now, $\forall n \geqslant N$, (By Theorem 9)

$$\int f_n dx \geqslant \int g_n dx$$

$$\geqslant \int_{B_n} g_n(x) dx$$

$$= \int_{B-(B-B_n)} g_n(x) dx$$

$$\geqslant \int_{B-(B-B_n)} (1-\xi)\varphi(x) dx$$

$$= (1-\xi)\left[\int_B \varphi(x)dx - \int_{B-B_n} \phi(x)dx\right]$$

$$\geqslant (1-\xi)\int \phi(x)dx - \int_{B-B_0} \varphi(x)dx$$

$$\geqslant \int \phi dx - \xi \int \varphi dx - m(B - B_n)\cdot M$$

$$\geqslant \int \phi dx - \xi \int \varphi dx - \varepsilon \cdot M$$

$$= \int \varphi dx - \xi\left[\int \varphi dx + M\right]$$

$\therefore \xi$ is arbitrary and M is finite

$$\int f_n dx \geqslant \int \varphi dx$$

(i.e.,) $\int \varphi dx \leqslant \int f_n dx$

$$\Rightarrow \int \varphi dx \leq \lim\inf \int f_n dx$$

$\therefore$ From case (i) & (ii), we get, $\int \phi dx \leqslant \lim \inf \int f_n dx$

$$\Rightarrow \sup_{\Phi \leqslant f} \int \Phi dx \leqslant \lim\inf \int f_n dx$$

$$\Rightarrow \int f dx \leqslant \lim \inf \int f_n dx$$

$$\Rightarrow \int \lim \inf f_n dx \leqslant \lim \inf \int f_n dx$$

Theorem 4 [Lebesgue's Monotone Convergence Theorem]

Let $\{f_n, n = 1,2, \ldots,$ be a sequence of non-negative measurable functions such that $\{f_n(x)\}$ is monotone increasing for each x. Let $f = \lim \mathrm{fn}_n$. Then $\int f dx = \lim \int f_n dx$.

Proof:

Let $\{f_n, n = 1,2, \ldots\}$ be a sequence of non-negative measurable functions.

Let $\{f_n(x)\}$ be a monotone increasing for each ' x '.

let $f = \lim f_n$ (i.e.,) $f_n \to f$

T.P: $\int f dx = \lim \int f_n dx$

Now, $\lim f_n = f \Rightarrow \lim\inf f_n = f$ ……..(1)

By Fatou's Lemma, we get.

$\int \lim\inf f_n dx \leqslant \lim\inf \int f_n dx$.

$$\Rightarrow \int f dx \leq \lim\inf \int f_n dx - \text{ (2) (by equation (1))}$$

Here f_n is increasing of $f_n \to f$

$$\therefore f_n \leqslant f \quad \forall n$$

$$\Rightarrow \int f_n dx \leqslant \int f dx$$

$$\Rightarrow \lim\sup \int f_n dx \leqslant \int f dx \ldots\ldots\ldots..(3)$$

From (2) of (3), we get,

$$\lim\sup \int f_n dx \leq \int f dx \leq \lim\inf \int f_n dx \leq \lim\sup \int f_A dx \leq \int f dx$$

$$\Rightarrow \int f dx = \lim\sup \int f_n dx = \lim\inf \int f_n dx$$

$$\Rightarrow \int f dx = \lim \int f_n dx$$

Theorem 5:

Let f be a non-negative measurable function Then there exists a sequence $\{\Phi_n\}$ of measurable simple functions such that, for each x , $\varphi_n(x) \uparrow f(x)$

Proof:

Let f be a non-negative measurable function.

We define the sequence $\{\varphi_A\}$ as follows:

Divide [0,1] in two equal portions

Let $E_{11} = \{x: 0 \leqslant f(x) \leqslant 1/2\}$

$E_{12} = \{x: 1/2 < f(x) \leqslant 1\}$

& $F_1 = \{x: f(x) > 1\}$

Let $\varphi_1 = 0x_{E_{11}} + 1/2x_{E_{12}} + 1x_{F_1}$

Divide [0,2] into 8 equal parts

Let $E_{21} = \{x: 0 \leqslant f(x) \leqslant 1/4\}$

$E_{22} = \left\{x: \frac{1}{4} < f(x) \leqslant \frac{1}{2}\right\}$

……….

………

$E_{28} = \{x: 7/4 < f(x) \leqslant 8/4\}$

& $F_2 = \{x: f(x) > 2\}$

Let $\varphi_2 = 0\chi_{E_{21}} + 1/4\chi_{E_{22}} + \cdots + 7/4x_{E_{28}} + 2x_{F_2}$

In general we divide $[0, n]$ into $n \cdot 2^{\mathrm{n}}$ equal intervals

Let $E_{n_k} = \left\{x: \frac{k-1}{2^n} < f(x) \leqslant \frac{k}{2^n}\right\}, k = 1,2, \dots, n2^n$ & $F_n = \{x: f(x) > n\}$

Let $\varphi_n = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} x_{E_{nk}} + nx_{f_n}$

Then the functions Φ_n are measurable simple functions Also $\varphi_n(x) \leqslant \varphi_{n+1}(x)$ for each ' x '.

If $f(x)$ is finite, then $x \in F_n{}^c$ ∀ large n '.

$\therefore |f(x) - \varphi_n(x)| \leqslant 2^{-n}$

$\therefore \varphi_n(x) \uparrow f(x)$

If $f(x) = \infty$, then $x \in \bigcap_{n=1}^{\infty} F_n$

$\therefore \varphi_n(x) = n \, \forall n$

$\therefore \varphi_n(x) \uparrow f(x)$

Corollary:

$\lim \int \varphi_n dx = \int f dx$ where f is a nonnegative measurable function &$\{\Phi_n\}$ is a sequence of measurable simple functions.

Proof:

By theorem $5, \{\varphi_n\}$ is a monotone increasing sequence.

$\therefore$ By theorem 4, $\int f dx = \lim \int \varphi_n dx$.

Theorem 6:

Let f and g be non-negative measurable functions. Then $\int f dx + \int g dx = \int (f+g) dx$.

Proof:

Let f & g be non-negative measurable functions.

Let Q & ψ be measurable simple functions

Let the values of φ be $a_1, a_2, \dots, a_n$ taken on sets $A_1, A_2, \dots, A_n$.

Let the values of ψ be $b_1, b_2, \dots, b_m$ taken on sets $B_1, B_2, \dots, B_m$.

Then the simple function $\phi + \psi$ has the value $a_i + b_j$ on the measurable set $A_i \cap B_j$

By Theorem 1 (i), we get, $\int_{A_i \cap B_j} (\phi + \psi) dx = \int_{A_i \cap B_j} \varphi dx + \int_{A_i \cap B_j} \psi dx$. ....(1)

But the union of n m disjoint sets $A_i \cap B_j$ is $\mathbb{R}$.

$$\therefore (1) \Rightarrow \int_{U(A_i \cap B_j)} (\phi + \psi) dx = \int_{\mathbb{R}} \varphi dx + \int_{U(\cap_i \cap B_j)} \psi dx.$$

$$\Rightarrow \int_{\mathbb{R}} (\varphi + \psi) dx = \int_{\mathbb{R}} \psi dx$$

$$\Rightarrow \int (\varphi + \psi) dx = \int \varphi dx + \int \psi dx \quad \dots\dots\dots\dots\dots \ (2)$$

Let $\{\Phi_n\}, \{\psi_n\}$ be sequences of measurable simple functions.

$\therefore$ By Theorem 5, $\Phi_n \uparrow f$ & $\psi_n \uparrow g. \Rightarrow \phi_n$ & $\psi_n \uparrow f + g$

By (2), $\int (\varphi_n + \psi_n) dx = \int \varphi_n dx + \int \psi_n dx$

letting $n \to \infty$ & By Theorem 4, we get $\int (f+g) dx = \int f dx + \int g dx$

Theorem 7:

Let $\{f_n\}$ be a sequence of non-negative measurable functions. Then $\int \sum_{n=1}^{\infty} f_n dx = \sum_{n=1}^{\infty} \int f_n dx$

Proof:

Let $\{f_n\}$ be a sequence of non-negative measurable functions.

By Theorem 6, we get, $\int (f+g) dx = \int f dx + \int g dx$ ....(1)

By induction, (1) applies to a sum of ' n ' functions.

Let $S_n = \sum_{i=1}^{n} f_i$

$\Rightarrow \int S_n dx = \int \sum_{i=1}^{n} f_i dx$

$= \int (f_1 + f_2 + \cdots + f_n) dx$

$= \int f_1 dx + \int f_2 dx + \cdots + \int f_n dx.$

$\therefore \int S_n dx = \sum_{i=1}^{n} \int f_i dx$

Let $f = \sum_{i=1}^{\infty} f_i$ ....(2)

Clearly $S_n \uparrow f$

$\therefore$ By Theorem 4, $\int f dx = \lim \int s_n dx.$

$\int \sum_{i=1}^{\infty} f_i dx = \lim \sum_{n=1}^{\infty} f_i dx = \sum_{i=1}^{\infty} \int f_i dx.$ (by (1) & (2)

$\therefore \int \sum_{n=1}^{\infty} f_n dx = \sum_{n=1}^{\infty} \int f_n dx.$

Example 4:

Give an example where strict inequality occurs in Fatou's Lemma.

Solution:

Let $f_{2n-1} = x_{[0,1]}, f_{2n} \equiv x_{(1,2)}, \; n = 1,2, ...$

Then $\lim\inf f_n(x) = 0 \quad \forall x$

$\Rightarrow \int \text{liminf} f \, f_n(x) dx = 0$

Also $\int f_n(x) dx = 1 \; \forall n$

$\Rightarrow \lim\inf f \int f_n(x) dx = 1$

$0 < 1$

$\int \lim\inf f_n(x) dx < \lim\inf \int f_n(x) dx$

Example 5:

Show that $\int_1^{\infty} \frac{dx}{x} = \infty$

The function x^{-1} is a continuous function for $x > 0 \therefore x^{-1}$ is measurable.

clearly it is positive

$\therefore$ The integral is defined also $\int_1^{\infty} \frac{dx}{x} > \int_1^{\infty} \frac{dx}{x}.$

But$_n$ $1/x > \frac{1}{k}$ $on\ [k-1,k)$

$$\therefore \int_1^n \frac{1}{x}dx > \sum_{k=2}^{n} \int_1^n \frac{1}{k} x_{(k-1,k)} dx$$

$$> \sum_{k=2}^{n} \frac{1}{k}$$

$$\to \infty \text{ as } n \to \infty$$

$$\int_1^\infty \frac{dx}{x} = \infty$$

Example 6:

$f(x)$,$0 \leqslant x \leqslant 1$, is defined by $f(x) = 0$ for x rational, if x is irrational, $f(x) = n$, where n is the number of zeros immediately after the decimal point, in the representation of x on the decimal scale. Show that f is measurable and find $\int_0^1 f dx$.

Solution:

Given: For $0 \leqslant x \leqslant 1, f(x) = \begin{cases} 0 & x \to \text{ rational} \\ n & x \to \text{ irrational} \end{cases}$

where $n \to$ No. of zeros immediately after the decimal point

For $x \in (0,1]$,

Let $g(x) = \begin{cases} 0 & x = 1 \\ n & 10^{-(n+1)} \leqslant x < 10^{-n}, n = 0,12 \end{cases}$

Then $f \leqslant g \Rightarrow f = ga \cdot e$ ……….(1)

Here ' g ' is measurable

∴ ' f ' also measurable.

By Example 3,

$$(1) \Rightarrow \int_0^1 f dx = \int_0^1 g dx$$

Now, $\int_0^1 g dx = \sum_{n=0}^{\infty} n \left(\frac{1}{10^n} - \frac{1}{10^n+1}\right)$

$$= \sum_{n=1}^{\infty} \frac{9n}{10^{n+1}} = \frac{9}{10} \sum_{n=1}^{\infty} \frac{n}{10^n} = \frac{1}{9}$$

$$\int_0^1 f dx = \frac{1}{9}$$

2.2 The General Integral:

Definition 4:

If $f(x)$ is any real function, $f^+(x) = \max(f(x), 0)\&$
$f^-(x) = \max(-f(x),0)$

are said to be the positive and negative parts of f , respectively.

Theorem 8:

(i) $f = f^+ - f^-; |f| = f^+ + f^-; f^+, f^- \geqslant 0$

(ii) f is measurable iff $f^+ + f^-$are both measurable.

Proof:

Let $f(x)$ be any real function

Let $f^+(x) = \max(f(x),0)$ & $f^{-1}(x) = \max(-f(x),0)$

$$= -\min(f(x),0)$$

(i)claim: $f^+, f^- \geqslant 0$

If $f(x) > 0$, then $f^+$is positive

If $f(x) < 0$, then $-f(x) > 0$

$\therefore$ $f^-$is positive

$\therefore$ $f^+, f^- \geqslant 0$

claim: $f = f^+ - f^- d|f| = f^+ + f^-$

We know that,

For any two functions f & $g, \max(f, g) = \frac{|f-g|+f+g}{2}$

Take $g = 0$.

$$\therefore f^+ = \max(f, 0) = [|f| + f]/2$$
$$\Rightarrow f^- = \max(-f, 0) = [1 - f\ | + (-f)]/2 = [|f| - f]/2$$
$$\therefore f^+ - f^- = \frac{|f| + f}{2} - \frac{|f| - f)}{2} = f(i), f^+ - f^- = f$$

$$f^+ + f^- = \frac{|f| + f}{2} + \frac{|f| - f}{2} = |f| \text{ (ie)}, f^+ + f^- = |f|$$

(ii) Suppose ' f ' is measurable.

We know that, The constant function ' O ' is measurable. Then sup $\{f, 0\}$ & inf $\{f, 0\}$ are measurable. $\therefore \max(f, 0) \& -\min(f, 0)$ are measurable (i.e.,) f^+ of $f^-$are measurable conversely, Suppose that f^t of $f^-$are measurable

Then $f^+ - f^-$is measurable

(i.e.,) f is measurable (by (i))

Definition 5:

If f is a measurable function and $\int f^+dx < \infty$, $\int f^-dx < \infty$, we say that f is integrable, and its integral is given by $\int \; fdx = \int \; f^+dx - \int \; f^-dx$

Clearly, a measurable function ' f ' is integrable inf $|f|$ is also measurable.

Also $$\int |f|dx = \int f^+dx + \int f^-dx$$

Definition 6:

If E is a measurable set, f is a measurable function, and $x_E f$ is integrable, we say that f is integrable over E, and its integral is given by $\int_E fdx = \int fx_E dx$. The notation $f \in L(E)$ is then sometimes used.

Definition 7:

If f is a measurable function such that at least one of $\int f^+dx, \int f^-dx$ is finite, then

$\int fdx = \int f^+dx - \int f^-dx$.

Note:

' f ' is said to be integrable only if the conditions of Definition ' 5 ' are satisfied, (i.e.,) if $|f|$ has a finite integral.

Theorem 9:

Let f & g be integrable functions.

(i) af is integrable and $\int$ af $dx = a\int fdx$.

(ii) $f + g$ is integrable, and $\int (f + g)dx = \int fdx + \int gdx$.

(iii) If $f = 0$ a.e, then $\int fdx = 0$

(iv) If $f \leqslant g$ a.e, then $\int fdx \leqslant \int gdx$

(v) If A and B are disjoint measurable sets, then $\int_A fdx + \int_B fdx = \int_{A\cup B} fdx$.

Proof:

Let f & g be integrable functions.

$$\Rightarrow \int f^+dx < \infty \;\; \& \int f^-dx < \infty \; \& \int fdx = \int f^+dx - \int f^-dx$$

Case (i): $a \geqslant 0$

$$(a_f)^+ = a_f^+ \qquad (a_f)^- = a_f^-$$
$$\Rightarrow \int (a_f)^+ dx < \infty \;\&\; \int (a_f)^- dx < \infty$$

∴ af is integrable

$$\int afdx = \int a_f^+ dx - \int a_f - dx$$
$$= a\left[\int f^+ dx - \int f^- dx\right]$$
$$= a\int fdx$$
$$\therefore \int a_f dx = a\int fdx.$$

case (ii): $a = -1$

$$\therefore (af)^+ = (-f)^+ = f^- \& \, (af)^- = (-f)^- = f^+$$
$$\Rightarrow \int f^- dx < \infty \,\& \int f^+ dx < \infty$$

∴ $af = -f$ is integrable.

$$\& \int afdx = \int af^+ dx - \int af^- dx$$
$$\Rightarrow \int (-f)dx = \int f^- dx - \int f^+ dx$$
$$= -\left[\int f^+ dx - \int f^- dx\right]$$
$$\int (-f)dx = -\int fdx$$

case (iii) : $a < 0$

$$af = -|a|f$$
$$\therefore \int afdx = -\int |a|fdx = -|a|\int fdx \text{ (by case (i))}$$
$$= a\int fdx$$

From case (i), (ii) & (iii) we get, $\int afdx = a\int fdx$

(ii) Now, We know that $(f+g)^+ \leqslant f^+ + g^+ + (f+g)^- \leqslant f^- + g^-$ $\because f+g$ are integrable, $\int (f+g)^t dx < \infty + \int (f+g)^- dx < \infty \therefore (f+g)$ is integrable.

Also, $(f+g)=(f+g)^+-(f+g)^-$
$\&(f+g)=f^+-f^-+g^+-g^-$
$\Rightarrow (f+g)^+-(f+g)^-=f^+-f^-+g^+-g^-$
$\Rightarrow (f+g)^++f^-+g^-=(f+g)^-+f^++g^+$

$$\Rightarrow \int [(f+g)^++f^-+g^-]dx=\int [(f+g)^-+f^++g^+]dx$$

$$\Rightarrow \int (f+g)^+dx+\int f^-dx+\int g^-dx=\int (f+g)^-dx+\int f^+dx+\int g^+dx$$

$$\Rightarrow \int (f+g)^+dx-\int (f+g)^-dx=\int f^+dx-\int f^-dx+\int g^+dx-\int g^-dx$$

$$\Rightarrow \int (f+g)dx=\int fdx+\int gdx.$$

(iii)

Given $f=0$ a.e
$\Rightarrow f^+=0$ a.e $f^-=0$ a.e

$\because f^+\& f^-$are non-negative measurable functions,

$\int f^+dx=0+\int f^-dx=0$ [By example :3]

$$\Rightarrow \int f^+dx-\int f^-dx=0$$

(i.e.,)$\int fdx=0$

(iv) Given: $f\leqslant g$ a.e

Let $g=f+(g-f)$

$$\int gdx=\int fdx+\int (g-f)^+dx-\int (g-f)^-dx$$

Here $(g-f)^-=0$ a.e

$\therefore \int (g-f)^-dx=0$ (by (iii))

$$\therefore \int gdx=\int fdx+\int (g-f)^+dx$$

$$\Rightarrow \int gdx\geqslant \int fdx$$

(i.e.,) $\int fdx\leqslant \int gdx$

(v) Let A & B be disjoint measurable sets.

Now, $\int fdx=\int f\chi_{A\cup B}dx$

$$A \cup B = \int f(X_A + \chi_B)dx \; [\because \chi_{A\cup B} = \chi_A + \chi_B]$$
$$= \int f x_A dx + \int f x_B dx$$
$$= \int_A f dx + \int_B f dx$$
$$\therefore \int_{A\cup B} f dx = \int_A f dx + \int_B f dx.$$

Note:

From theorem 9 , if $f = g$ a.e and f & g are integrable, then $\int f dx = \int g dx$.

We can extend our results to the case where f is measurable and f is defined except on the set E such that $m(E) = 0$ and $\int_{EC} |f| dx < \infty$. Then we define f arbitrarily on E to get a function g which clearly is necessarily integrable.

Example 1:

Show that if f & g are measurable, $|f| \leqslant |g|$ a.e and g is integrable, then f is integrable.

Solution:

Let f & g be measurable.

Let $|f| \leqslant |g| a \cdot e$ & g is integrable

To prove: f is integrable.

Redefine ' f ' on a set of measure zero.

suppose $|f| \leqslant |g|$

$$\Rightarrow f^+ \leqslant |g| \text{ \\& } f^- \leq |g|$$
$$\Rightarrow \int f^+ dx \leqslant \int |g| dx \; \& \int f^- dx \leqslant \int |g| dx$$

$\because$ g is integrable, $\int |g| dx < \infty$

$$\therefore \int f^+ dx < \infty \text{ of } \int f^- dx < \infty$$

$\therefore$ f is integrable.

Example: 8

Show that if f is an integrable function, then $\left|\int f dx\right| \leqslant \int |f| dx$. When does equality occur?

Solution:

Given: f is an integrable function.

We know that $|f| \geqslant f$

$\Rightarrow |f| - f \geqslant 0$

$\Rightarrow \int |f| - f) dx \geqslant 0$

$\Rightarrow \int |f| dx - \int f dx \geqslant 0$

$\Rightarrow \int |f| dx \geqslant \int f dx \quad \ldots\ldots\ldots..(1)$

Also $|f| \geqslant -f$

$\Rightarrow |f| + f \geqslant 0$

$\Rightarrow \int (|f| + f) dx \geqslant 0$

$\Rightarrow \int |f| dx + \int f dx \geqslant 0$

$\Rightarrow \int |f| dx \geqslant -\int f dx \quad \ldots\ldots\ldots.(2)$

∴ From (1) & (2),

$\int |f| dx \geqslant |\int f dx|$ …………. (3)

Necessary Condition for Equality:

if $\int f dx \geqslant 0$, then $\int |f| dx = \int f dx$

$\Rightarrow \int (|f| - f) dx = 0$

$\Rightarrow |f| - f = 0$ a.e $[By$ Example 3]

$\Rightarrow |f| = f$ a.e

If $\int f dx < 0$, then $\int |f| dx = \int (-f) dx$

$\Rightarrow \int (|f| + f) dx = 0$

$\Rightarrow |f| + f = 0$ ale

$\Rightarrow |f| = -f a \cdot e$

From (4) & (5), we get $\int f dx \geqslant 0 \Rightarrow |f| = f$ a.e

& $\int f dx < 0 \Rightarrow |f| = -f\ a \cdot e$

$\Rightarrow \left|\int f dx\right| = \int |f| dx$ only when $f \geqslant 0$ abe (or) $f \leqslant 0$ a.e

Example 9:

If ' f ' is measurable and g integrable and α, β are real numbers such that $\alpha \leq f \leq \beta$ ale., then there exists $\gamma, \alpha \leqslant \gamma \leqslant \beta$ such that $\int f|g| dx = \gamma \int |\text{g}| dx$

Solution:

Let ' f ' be measurable & ' g ' be integrable. Let α, β be real numbers 7: $\alpha \leqslant f \leqslant \beta$ a.e.

To Prove: $\exists \gamma, \alpha \leqslant v \leqslant \beta \ \ni: \int f|g|dx = \gamma \int |g|dx$.

$$|fg| = |f| \cdot |g|$$

Now, $\leqslant (|\alpha| + |\beta|) \cdot |g| \quad a \cdot e$

$$\therefore |fg| \leqslant (|\alpha| + |\beta|) \cdot |g| \quad a \cdot e$$

$\Rightarrow fg$ is measurable (by Example:7)

Also $\alpha \leqslant f \leqslant \beta$ a.e

$$\Rightarrow \alpha|g| \leqslant f|g| \leqslant \beta|g| \quad a \cdot e$$

$$\Rightarrow \alpha \int |g|dx \leqslant \int f|g|dx \leqslant \beta \int |g|dx$$

if $\int |g|dx = 0$, then

$g = 0$ a.e

$$\therefore \int f|g|dx = 0 = \gamma \int |g|dx$$

$if \int |g|dx \neq 0$, then

Take $\gamma = \left(\int f|g|dx\right)\left(\int |g|dx\right)^{-1}$

$$\Rightarrow \int f|g|dx = \gamma \int |g|dx$$

Example 10:

Extend Theorem 9 to any functions such that the integral involved are defined in the sense of Definition 7. (i)$\int fdx = \int f^+dx - \int f^-dx$.

Solution:

We consider, for example, the extension of (ii): If $\int (f+g)dx, \int fdx + \int gdx$ are defined, then $\int (f+g)dx = \int fdx + \int gdx$ whenever the right hand side is defined.

To prove this, suppose $\int fdx = \infty = \int gdx$

Then $\int f^-dx < \infty$ & $\int g^-dx < \infty$

$$\Rightarrow \int (f+g)^{-}dx < \infty \Rightarrow \int f(f+g)dx = \infty$$

We know that $(f+g)^{+} + f^{-} + g^{-} = (f+g)^{-} + f^{+} + g^{+}$

$$\Rightarrow \int (f+g)^{+} - \int (f+g)^{-} = \int f + dx - \int f - dx + \int g + dx - \int g - dx$$

$$\Rightarrow \int (f+g)dx = \int fdx + \int gdx$$

$$\Rightarrow \infty = \infty$$

The same argument works if $\int fdx < \infty + \left|\int gdx\right| < \infty$.

Example 11:

Show that if f is integrable, then f is finite valued a.e.

Solution:

Let f be integrable. Suppose $|f| = \infty$ on a set E with $m(E) > 0$

$$\int |f|dx > n\, m(E) \ \forall \text{n}$$

$\Rightarrow\Leftarrow$ (f integrable $\Rightarrow \int |fdx| < \infty$). $\therefore f$ is finite valued a.e.

Example 12:

If f is measurable, $m(E) < \infty$ and $A \leq f \leq B$ on E, then $Am(E) \leqslant \int_E fdx \leqslant B \cdot m(E)$

Solution:

Let f be measurable, $M(E) < \infty$ of $A \leqslant f \leqslant B$ on E.

To prove: $Am(E) \leqslant \int_E fdx \leqslant B \cdot m(E)$

Now,

$A \leq f \leq B \Rightarrow A\chi_E \leqslant f\chi_E \leqslant B\chi_E$

$$\Rightarrow \int Ax_E dx \leqslant \int f\chi_E dx \leqslant \int Bx_E dx$$

$$\Rightarrow Am(E) \leqslant \int_E fdx \leqslant B \cdot m(E).$$

Theorem 10 [Lebesgue's Dominated Convergence Theorem]:

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leqslant g$, where g is integrable, and let $\lim f_n = f$ a.e. Then f is integrable and $\lim \int f_n dx = \int fdx$.

Proof:

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leqslant g$.

Let g be integrable.

Let $\lim f_n = f$ a.e.

To prove: f is integrable of $\lim_{\int} \int dx = \int f dx$.

Here each f_n is measurable

Then f is measurable.

Also for each n, $|f_n| \leq$ g

$\Rightarrow |f| < g$ a.e.

Bu Example 7, f_n & f are integrable

To Prove: $\lim \int f_n dx = \int f dx$.

(i.e.,)To prove lim inf $\int f_n dx \geqslant \int f dx$ & lim sup $\int f_n dx \leqslant \int f dx$

Now,

$$|f_n| \leqslant g \Rightarrow -g \leqslant f_n \leqslant g$$
$$\Rightarrow -g \leqslant f_n \text{ \& } f_n \leqslant g$$
$$\Rightarrow g + f_n \geqslant 0 \text{ \& } g - f_n \geqslant 0 \quad \ldots\ldots\ldots (1)$$

Now, $g + f_n \geqslant 0$

$\therefore \{g + f_n\}$ is a sequence of non-negative measurably functions.

By Fatou's Lemma, we get,

$$\int \liminf (g + f_n) dx \leqslant \liminf \int (g + f_n) dx$$
$$\Rightarrow \int g dx + \int \liminf f_n dx \leqslant \int g dx + \liminf \int f_n dx$$

($\because g$ is independent of n)

$\Rightarrow \int \liminf f_n dx \leqslant \liminf \int f_n dx$ ($\because \int g dx$ is finite)

$\Rightarrow \int f dx \leqslant \liminf \int f_n dx$ Le) ……..(2) $[\because \lim f_n = f a \cdot e]$

(1) $\Rightarrow$ $g - f_n \geqslant 0$

$\therefore \{g - f_n\}$ is a sequence of non-negative measurable functions.

By Fatou's Lemma, we get,

$$\int \liminf(g - f_n)dx \leqslant \liminf \int (g - f_n)dx$$

$$\Rightarrow \int gdx + \int \liminf(-f_n)dx \leqslant \int gdx + \liminf \int (-f_n)dx$$

($\because$ g is independent of n)

$$\Rightarrow \int \liminf(-f_n)dx \leqslant \liminf \int (-f_n)dx \left(\because \int gdx \to \text{finite}\right)$$

$$\Rightarrow -\int \limsup f_n dx \leqslant -\limsup \int f_n dx$$

$$\Rightarrow \int \limsup f_n dx \geqslant \limsup \int f_n dx$$

$$\Rightarrow \int fdx \geqslant \limsup \int f_n dx$$

Enron (2) & (3) we get,

$$\limsup \int f_n dx \leqslant \int fdx \leqslant \liminf \int f_n dx \leqslant \limsup \int f_n \mathrm{dx}$$

$$\Rightarrow \int fdx = \limsup \int f_n dx = \liminf \int f_n dx.$$

$$\Rightarrow \int fdx = \lim \int f_n dx.$$

Example 13:

With the same hypotheses as Theorem 10, show that $\lim \int |f_n - f|dx = 0$

(i.e.,)Let $\{f_n\}$ be a sequence of measurable functions such that $|fn| \leqslant g$, where g is integrable, and let $\lim f_n = f$ ae. Then $\lim \int |f_n - f|dx = 0$.

Solution:

Now, $|f_n - f| \leqslant |f_n| + |f| \leqslant g + g = 2g$

(i.e.,) $|fn - f| \leqslant 2g \qquad \forall n$

$\therefore$ $\lim f_n = f a.e$, $\lim |f_n - f| = 0$ a.e.

$\therefore$ By Theorem 10 to $\{f_n - f\}$, we get, $\lim \int |f_n - f|dx = 0$

Theorem 11:

Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n|dx < \infty$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. its sum $f(x)$ is integrable and $\int fdx = \sum_{n=1}^{\infty} \int f_n dx$.

Proof:

Let $\{fn\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n|dx < \infty$

To prove:(i) $\sum_{n=1}^{\infty} f_n(x)$ converges a.e

(ii) $f(x) = \sum_{n=1}^{a} f_n(x)$ is integrable.

(iii) $\int f dx = \sum_{n=1}^{\infty} \int f_n dx$.

Let $\varphi(x) = \sum_{n=1}^{\infty} |f_n|$

clearly, $|f_n|$ be a sequence of non-negative measurable,

Then by theorem 7,

$$\int \sum_{n=1}^{\infty} |f_n| dx = \sum_{n=1}^{\infty} \int |f_n| dx$$

(i.e.,) $\therefore \varphi(x)dx = \sum_{n=1}^{\infty} \int |f_n| dx < \infty$ ($\because f_n$ is integrable $\Rightarrow \int |f_n| < \infty$)

(i), $\int \varphi(x)dx < \infty$

$\Rightarrow$ $'\varphi'$ ' is integrable

$\therefore$ Φ is a finite valued ace (By Example 11)

Also $f = \sum_{n=1}^{\infty} f_n(x)$ & $\phi = \sum_{n=1}^{\infty} |f_n(x)|$,

$$\Rightarrow |f| \leqslant \varphi$$

$$\Rightarrow \int |f| dx < \int \varphi dx < \infty$$

$\therefore$ f is integrable.

Let $g_n(x) = \sum_{i=1}^{n} f_i(x)$

$\Rightarrow |g_n(x)| \leqslant \varphi(x)d$

$\lim g_n(x) = f(x)$ ae

$\therefore \lim \int g_n(x)dx = \int f(x)dx$ [by Theorem 10]

$$\Rightarrow \sum_{n=1}^{\infty} \int f_n dx = \int f dx$$

Example 14:

In Theorems 10 and 11 we may suppose that the hypotheses hold only on a measurable set E. Then theorem 10 and example 13, with internals taken over E, follow on replacing throughout f_n, f etc., by $f_n \chi_E, f \chi_E$, etc.

Example 15:

Theorem 10 deals with a sequence of functions $\{f_n\}$. State and prove a 'continuous parameter' version of the theorem.

Solution:

Theorem: For each $\xi \in [a,b], -\infty \leqslant a < b < \infty$, let f_ξ be a measurable function, $|f_\xi(x)| \leqslant g(x)$ where g is an integrable function, and let $\lim_{\xi \to \xi_0} f_\xi(x) = f(x)$ a.e., where $\xi_0 \in [a,b]$. Then f is integrable and $\lim_{\xi \to \xi_0} \int f_\xi \mathrm{d}x = \int f \, \mathrm{d}x$

Proof:

Let $\{\xi_n\}$ be any sequence in $[a,b], \lim \xi_n = \xi_0$. Then the sequence $\{f_{\xi_n}\}$ satisfies the conditions of Theorem 10, and we deduce that f is integrable. Suppose that (3.15) does not hold. Then $\exists \delta > 0$ and a sequence $\{\beta_n\}$, with $\lim \beta_n = \xi_0$, such that for all $n, \left|\int f_{\beta_n} \, \mathrm{d}x - \int f \, \mathrm{d}x\right| > \delta$. But, applying Theorem 10 to the sequence $\{f_{\beta_n}\}$, we get a contradiction.

Example 16:

(i) If f is integrable, then $\int f \, \mathrm{d}x = \lim_{a \to \infty} \lim_{b \to -\infty} \int_b^a f \, \mathrm{d}x = \lim_{b \to -\infty} \lim_{a \to \infty} \int_b^a f \, \mathrm{d}x$.

(ii) If f is integrable on $[a,b]$ and $0 < \epsilon < b - a$, then

$$\int_a^b f \, \mathrm{d}x = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f \, \mathrm{d}x$$

Solution:

$\int_b^a f \, \mathrm{d}x = \int_{-\infty}^a \chi_{[b,\infty)} f \, \mathrm{d}x$. (by Example 15)

$$\lim_{b \to -\infty} \int_{-\infty}^a \chi_{[b,\infty)} f \, \mathrm{d}x = \int_{-\infty}^a f \, \mathrm{d}x$$

A second application of Example 15 gives the first equation of and the second follows in the same way; (ii) is proved similarly.

The following theorem, which will be generalized in Theorem 9, allows us to calculate integrals in many cases of importance.

Theorem 12:

If f is continuous on the finite interval $[a,b]$, then f is integrable, and $F(x) = \int_a^x f(t)\mathrm{d}t (a < x < b)$ is a differentiable function such that $F'(x) = f(x)$.

Proof:

As f is continuous, it is measurable and $|f|$ is bounded. So f is integrable on $[a,b]$. If $a < x < b$ we have $x + h \in (a,b)$ for all small h, and $F(x+h) - F(x) = \int_x^{x+h} f(t)\mathrm{d}t$.

But using Example 12 and the continuity of f we have

$$\int_x^{x+h} f(t)\mathrm{d}t = hf(\xi), \xi = x + \theta h, 0 \leqslant \theta \leqslant 1.$$

So, supposing $h \neq 0$, dividing by h and letting $h \to 0$, we get the result.

Corollary 1:

Integrals of elementary continuous functions over finite intervals can be calculated in the usual way using indefinite integrals.

Corollary 2:

From Example 16 it follows that the integral of an integrable continuous function over an infinite interval can be obtained if its indefinite integral is known.

Corollary 3:

Techniques involving integration by parts and by substitution can be employed if all the functions involved are continuous and integrable. Infinite intervals can be dealt with in this case as in Example 16.

Corollary 4:

In the case of piecewise-continuous functions, if we split the domain appropriately, we can calculate the separate integrals as in Corollary 1.

Using Theorem 12 and its corollaries we can now give specific examples which show some ways in which Lebesgue's Dominated Convergence Theorem (Theorem 10) may be used.

Example 17:

Show that if $\alpha > 1$, $\int_0^1 \frac{x\sin x}{1+(nx)^\alpha}\mathrm{d}x = o(n^{-1})$ as $n \to \infty$.

Solution:

We wish to show that $\lim\limits_{n\to\infty} \int_0^1 \frac{nx\sin x}{1+(nx)^\alpha}\mathrm{d}x = 0$

Clearly $\lim_{n\to\infty} \frac{nx\sin x}{1+(nx)^\alpha} = 0$, so we wish to show that Theorem 10 applies to the sequence

$$f_n(x) = \frac{nx\sin x}{1+(nx)^\alpha}, n = 1,2, \ldots$$

We consider $h(x) = 1 + (nx)^{\alpha} - nx^{3/2}$. So $h(0) = 1, h(1) = 1 + n^{\alpha} - n$. For $1 < \alpha \leqslant 3/2, h$ has no stationary point in [0,1], for all large n; for $\alpha > 3/2$ it has a stationary point at which its value is easily seen to approach 1 for large n. It follows that for large $n, h(x) > 0$ on [0,1] and so $\left|\frac{nx\sin x}{1+(nx)^{\alpha}}\right| \leqslant \frac{1}{\sqrt{1x}}$ and the result follows.

Example 18:

Show that $\lim \int_0^{\infty} \frac{dx}{(1+x\ln)^n x^{1/n}} = 1$

Solution:

For $n > 1, x > 0$, $(1 + x/n)^n = 1 + x + \frac{n(n-1)}{n^2}\frac{x^2}{2} + \cdots > \frac{x^2}{4}$

So if we define $g(x) = 4/x^2 (x \geqslant 1), g(x) = x^{-1/2}(0 < x < 1)$ we have

$(1 + x/n)^{-n} x^{-1/m} < g(x), (n > 1, x > 0)$.

But g is integrable over $(0, \infty)$, so $\lim \int_0^{\infty} (1 + x/n)^{-n} x^{-1/m}\, \mathrm{d}x = \int_0^{\infty} e^{-x}\, \mathrm{d}x = 1$.

Example 19:

Show that $\lim_{n\to\infty} \int_a^{\infty} \frac{n^2 x e^{-n^2x^2}}{1+x^2}\, \mathrm{d}x = 0$ for $a > 0$, but not for $a = 0$.

Solution:

If $a > 0$, substitute $u = nx$ to get

$$\int_a^{\infty} f_n(x)\mathrm{d}x = \int_{na}^{\infty} \frac{ue^{-u^2}}{1 + u^2/n^2}\, \mathrm{d}u = \int_0^{\infty} \chi_{(na,\infty)} \frac{ue^{-u^2}}{1 + u^2/n^2}\, \mathrm{d}u,$$

and the last integrand is less than ue^{-u^2}, an integrable function. But, as $a > 0$, $\lim_{i\to\infty} \chi_{(na,\infty)} (1 + u^2/n^2)^{-1} ue^{-u^2} = 0$. So Theorem 10 gives the result.

If $a = 0$, the same substitution gives

$$\int_0^{\infty} f_n(x)\mathrm{d}x = \int_0^{\infty} ue^{-u^2}(1 + u^2/n^2)^{-1}\, \mathrm{d}u \to \int_0^{\infty} ue^{-u^2}\, \mathrm{d}u = 1/2$$

using Theorem 10.

Example 20:

Let f be a non-negative integrable function on $[0,1]$. Then there exists a measurable function $\varphi(x)$ such that φf is integrable on $[0,1]$ and $\varphi(0+) = \infty$.

Solution:

It follows easily from Example 15 that $\lim_{a\to 0}\int_0^a f\,\mathrm{d}x = 0$. So $\forall n, \exists x_n\ (0 < x_n < 1)$, such that $\int_0^{x_n} f\,\mathrm{d}x < n^{-3}$, and we may suppose that $x_n \downarrow 0$ as $n \to \infty$.

Define $\varphi(x) = \sum_{k=2}^{\infty}(k-1)\chi_{(x_k,x_{k-1}]}$. So $\varphi(0+) = \infty$.

But $\int_{x_k}^{x_{k-1}} \varphi f\,\mathrm{d}x = \int_{x_k}^{x_{k-1}}(k-1)f\,\mathrm{d}x < (k-1)^{-2}$. So $\int_0^1 \varphi f\,\mathrm{d}x \leqslant \sum_{n=1}^{\infty} 1/n^2 < \infty$.

2.3. Riemann and Lebesgue Integrals

We consider the Riemann integral of a bounded function f over a finite interval $[a,b]$.

Let $a = \xi_0 < \xi_1 < \cdots < \xi_n = b$ be a partition, D, of $[a,b]$. Write $S_D = \sum_{i=1}^{n} M_i(\xi_i - \xi_{i-1})$ where $M_i = \sup f$ in $[\xi_{i-1},\xi_i], i = 1,\dots,n$. Similarly on replacing M_i by m_i equal to inf f over the corresponding interval, we obtain $s_D = \sum_{i=1}^{n} m_i(\xi_i - \xi_{i-1})$. Then f is said to be Riemann integrable over $[a,b]$ if given $\epsilon > 0$, there exists D such that $S_D - s_D < \epsilon$. In this case we have inf $S_D = \sup s_D$, where the infimum and supremum are taken over all partitions D of $[a,b]$, and we write the common value as $\mathrm{R}\int_a^b f\,\mathrm{d}x$.

Theorem 13:

If f is Riemann integrable and bounded over the finite interval $[a,b]$, then f is integrable and $\mathrm{R}\int_a^b f\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x$.

Proof:

Let $\{D_n\}$ be a sequence of partitions such that, for each $n, S_{D_n} - s_{D_n} < 1/n$. It is easily seen that

$$S_{D_n} = \int_a^b u_n\,\mathrm{d}x \text{ and } s_{D_n} = \int_a^b l_n\,\mathrm{d}x$$

where u_n and l_n are step functions, $u_n \geqslant f \geqslant l_n$. Indeed we may, for example, define $u_n = M_i$ on (ξ_{i-1},ξ_i), and at a partition point let u_n be the average of the values M_i corresponding to the intervals ending at that point. Write $U = \inf_n u_n$ and $L = \sup_n l_n$. Now

$$[x: U(x) - L(x) > 0] = \bigcup_{k=1}^{\infty} [x: U(x) - L(x) > 1/k]$$

But if $U - L > 1/k$, then $u_n - l_n > 1/k$ for each n. So if $m[x: U(x) - L(x) > 1/k] = a$, then $\int (u_n - l_n)\mathrm{d}x > a/k$, and so $a/k < 1/n$ for each n. So $a = 0$. Hence $U - L \leqslant 1/k$ a.e. for each k, so $U = L$ a.e.

But u_n, l_n and hence U, L are measurable. Also $L \leqslant f \leqslant U$, so f is measurable and, being bounded, is integrable. Clearly

$\int_a^b l_n \, \mathrm{d}x \leqslant \int_a^b f \, \mathrm{d}x \leqslant \int_a^b u_n \, \mathrm{d}x$

and letting $n \to \infty$, we get R $\int_a^b f \, \mathrm{d}x = \int_a^b f \, \mathrm{d}x$.

Note:

The converse does not hold. Consider for example the function f on $[0,1]$:

$$f(x) = \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational.} \end{cases}$$

Then f is measurable, indeed $f = 1$ a.e. So $\int_0^1 f \, \mathrm{d}x = 1$. But each $S_D = 1$ and each $s_D = 0$, so f is not Riemann integrable.

That the function f of this example is not Riemann integrable can be seen also from the next theorem, since f is discontinuous at each x in $[0,1]$. The theorem shows that the class of Riemann-integrable functions is quite restricted.

Theorem 14:

Let f be a bounded function defined on the finite interval $[a, b]$, then f is Riemann integrable over $[a, b]$ if, and only if, it is continuous a.e.

Proof:

Suppose that f is Riemann integrable over $[a, b]$. Using the notation of the last theorem, suppose that $U(x) = f(x) = L(x)$, where x is not a partition point of any D_n, the D_n being chosen as before. Then f is continuous at x; for otherwise there would exist $\epsilon > 0$ and a sequence $(x_k\}, \lim x_k = x$, such that for each $k, |f(x_k) - f(x)| > \epsilon$. But then $U(x) \geqslant L(x) + \epsilon$. Now, the set of all partition points of the D_n is countable and so has measure zero, and the set $[x: U(x) \neq L(x)]$ has measure zero by the proof of the last theorem. So f is continuous a.e.

Conversely, suppose that f is continuous a.e. Choose a sequence $\{D_n\}$ of partitions of $[a, b]$ such that, for each n, D_{n+1} contains the partition points of D_n and such that the length of the largest interval of D_n tends to zero as $n \to \infty$. Then if u_n, l_n are the corresponding step functions as in the last theorem, we have $u_{n+1} \leqslant u_n$ and $l_{n+1} \geqslant l_n$ for each n. Write $U = \lim u_n$ and $L = \lim l_n$. Now suppose that f is continuous at x.

Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $\sup f - \inf f < \epsilon$, where the supremum and infimum are taken over $(x-\delta, x+\delta)$. For all n sufficiently large, an interval of D_n containing x will lie in $(x-\delta, x+\delta)$, and so $u_n(x) - l_n(x) < \epsilon$. But ϵ is arbitrary so $U(x) = L(x)$. So $U = L$ a.e. But then, by Theorem 10,

$\lim \int \ u_n \, \mathrm{d}x = \int \ U \, \mathrm{d}x = \int \ L \, \mathrm{d}x = \lim \int \ l_n \, \mathrm{d}x$ and so f is Riemann integrable.

Definition 8:

If, for each a and b, f is bounded and Riemann integrable on $[a,b]$ and $\lim\limits_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f \, \mathrm{d}x$

exists, then f is said to be Riemann integrable on $(-\infty, \infty)$, and the integral is written $\mathrm{R}\int_{-\infty}^{\infty} f \mathrm{d}x$.

Theorem 15:

Let f be bounded and let f and $|f|$ be Riemann integrable on $(-\infty, \infty)$. Then f is integrable and

$$\int_{-\infty}^{\infty} f \, \mathrm{d}x = \mathrm{R}\int_{-\infty}^{\infty} f \, \mathrm{d}x$$

Proof:

From Theorem 13, $\int_a^b |f| \mathrm{d}x = \mathrm{R}\int_a^b |f| \mathrm{d}x \leqslant \mathrm{R}\int_{-\infty}^{\infty} |f| \mathrm{d}x$.

for all a and b. So f is integrable. Theorem 13, applied again, gives $\int_a^b f \, \mathrm{d}x = \mathrm{R}\int_a^b f \, \mathrm{d}x$ and Example 16, gives the result.

The next result may be used to reduce problems involving integrals of measurable functions to more amenable classes of functions.

Theorem 16:

Let f be bounded and measurable on a finite interval $[a,b]$ and let $\epsilon > 0$. Then there exist

(i) a step function h such that $\int_a^b |f-h| dx < \epsilon$,

(ii) a continuous function g such that g vanishes outside a finite interval and $\int_a^b |f-g| dx < \epsilon$

Proof:

(i) As $f = f^+ - f^-$, we may assume throughout that $f \geqslant 0$. Now $\int_a^b f \, \mathrm{d}x = \sup \int_a^b \varphi \mathrm{d}x$, where $\varphi \leqslant f, \varphi$ simple and measurable. So we may assume that f is a simple measurable function, with $f = 0$ outside $[a,b]$. So $f = \sum_{i=1}^{n} \ a_i \chi_{E_i}$

with $\cup_{i=1}^{n} E_i = [a,b]$. Let $\epsilon' = \epsilon/nM$ where $M = \sup f$ on $[a,b]$, and M may obviously be supposed positive. For each of the measurable sets E_i there exist open intervals $I_1, \dots, I_k$ such that, if $G = \cup_{r=1}^{k} I_r$, then $m(E_i \Delta G) < \epsilon'$. But χ_G is a step function such that $\int |\chi_{E_i} - \chi_G| dx = m(E_i \Delta G) < \epsilon'$. Construct such step functions h_i, say, for each E_i,

Then $\int_a^b |f - \sum_{i=1}^{n} a_i h_i| \mathrm{d}x < \sum_{i=1}^{n} a_i \epsilon' \leqslant nM\epsilon' = \epsilon$ But $\sum_{i=1}^{n} a_i h_i$ is a step function.

(ii) From (i) there exists a step function h vanishing outside a finite interval (note that this interval need not be identical with $[a,b]$), such that $\int_a^b |f - h| \mathrm{d}x < \epsilon/2$

The proof is completed by constructing a continuous function g such that $\int |h - g| \mathrm{d}x < \epsilon/2$ and such that $g(x) = 0$ whenever $h(x) = 0$. Let $h = \sum_{i=1}^{n} a_i \chi_{E_i}$ where E_i is the finite interval $(c_i, d_i), i = 1, \dots, n$. As in (i), it is sufficient to show that each χ_{E_i} may be approximated. We may suppose that $\epsilon < 2(d_i - c_i)$ and define g by: $g = 1$ on $(c_i + \epsilon/4, d_i - \epsilon/4), g = 0$ on $\mathbf{C}(c_i, d_i)$. Extend g by linearity to $(c_i, c_i + \epsilon/4)$ and $(d_i - \epsilon/4, d_i)$, as in Fig. 2.1, to get a continuous function. Clearly $\int |\chi_{E_i} - g| dx < \epsilon/2$, and (ii) follows.

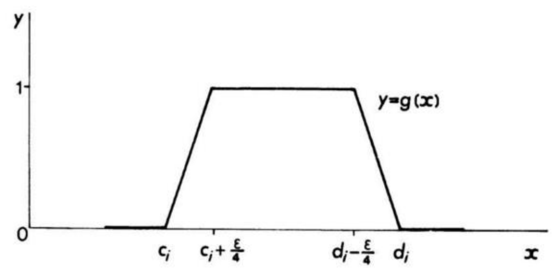

Figure 2.1

Corollary:

The results of Theorem 16 hold if f is integrable over $[a,b]$, using Exercise 4, p. 60, since, as in the proof, we may assume $f \geqslant 0$.

Example 24:

Let f be a bounded measurable function defined on the finite interval(a,b). Show that $\lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x \, \mathrm{d}x = 0$.

Solution:

By Theorem 16, $\forall \epsilon > 0, \exists h = \sum_{i=1}^{n} \xi_i \chi_{(a_i, b_i)}$, say, with $\int_a^b |f - h| dx < \epsilon$. Then

$$\left|\int_a^b f\sin\beta x dx\right| \leqslant \int_a^b |(f-h)\sin\beta x| dx + \left|\int_a^b h\sin\beta x dx\right|$$
$$< \epsilon + \left|\int_a^b h\sin\beta x dx\right|.$$

Now $\left|\int_a^b \chi_{(a_i,b_i)}\sin\beta x \, \mathrm{d}x\right| = \left|1/\beta\int_{\beta a_i}^{\beta b_i} \sin y \, \mathrm{d}y\right| \leqslant 2/\beta < \epsilon/nM$ for $\beta > \beta_0$, say, where $M = \max[\xi_i, i = 1, \ldots, n]$. So $\left|\int_a^b f\sin\beta x \, \mathrm{d}x\right| < 2\epsilon$, for $\beta > \beta_0$.

Example 25:

Show that if $f \in L(a+h, b+h)$ and $f_h(x) \equiv f(x+h)$, then $f_h \in L(a,b)$ and $\int_{a+h}^{b+h} f \, \mathrm{d}x = \int_a^b f_h \, \mathrm{d}x$.

Solution:

Clearly $(f_h)^+ = (f^+)_h, (f_h)^- = (f^-)_h$, so it is sufficient to prove the result for $f \geqslant 0$. By the corollary to Theorem 5,there exists a sequence of measurable simple functions $\{\varphi_n\}$ such that $\varphi_n \leqslant f$ and $\int \varphi_n \, \mathrm{d}x \uparrow \int f \, \mathrm{d}x$. But then $(\varphi_n)_h \uparrow f_h$, and so by monotone convergence

$$\int_{a+h}^{b+h} f \, \mathrm{d}x = \lim \int_{a+h}^{b+h} \varphi_n \, \mathrm{d}x = \lim \int_a^b (\varphi_n)_h \, \mathrm{d}x = \int_a^b f_h \, \mathrm{d}x.$$

UNIT III

Fourier Series and Fourier Integrals - Introduction - Orthogonal system of functions - The theorem on best approximation - The Fourier series of a function relative to an orthonormal system - Properties of Fourier Coefficients - The Riesz-Fischer Theorem - The convergence and representation problems in for trigonometric series - The Riemann - Lebesgue Lemma - The Dirichlet Integrals - An integral representation for the partial sums of Fourier series - Riemann's localization theorem - Sufficient conditions for convergence of a Fourier series at a particular point –Cesaro Summability of Fourier series- Consequences of Fejer's theorem - The Weierstrass approximation theorem

Chapter 3: Sections 3.1 to 3.14

Fourier Series and Fourier Integrals

3.1. Orthogonal system of functions:

Definition:

Let $S = \{\varphi_0, \varphi_1, \varphi_2, \dots\}$ be a collection of functions in $L^2(I)$. If $(\varphi_n, \varphi_m) = 0$ whenever $m \neq n$, the collection S is said to an orthogonal system on I. If, in addition, each φ_π has norm $\mathbf{1}^1$, then s is said to be orthonormal on I.

Note :

We denote $L^2(I)$ the set of all complexed valued functions which are measurable on I and R such that $|f|^2 \in L^2(I)$ the inner product of (f, g) of two suck function defined by $(f, g) = \int f(x)\overline{g(x)}dx$ always exists, then the non- negative number $\| f \| = (f, g)^{1/2}$ is the L^2 norm of f.

To Verify $s = \{\phi_0, \phi_1, \phi_2, \dots\}$ is orthonormal on I.

Let $S = \{\phi_0, \phi_1, \phi_2, \dots\}$ is orthonormal on I.

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}} \qquad \phi_1(x) = \frac{\cos x}{\sqrt{\pi}}, \phi_2(x) = \frac{\sin x}{\sqrt{\pi}}$$

$$I = [0, 2\pi] \qquad \phi_3(x) = \frac{\cos 2x}{\sqrt{\pi}}, \phi_4(x) = \frac{\sin 2x}{\sqrt{\pi}}$$

$$\vdots$$

$$\phi_{2\pi-1}(x) = \frac{\cos \pi x}{\sqrt{\pi}}, \phi_{2\pi}(x) = \frac{\sin \pi x}{\sqrt{\pi}}$$

$$(\phi_1,\phi_2) = \int_0^{2\pi} \phi_1(x)\overline{\phi_2(x)}dx$$

$$= \int_0^{2\pi} \frac{\cos x}{\sqrt{\pi}} \cdot \frac{\sin x}{\sqrt{\pi}} dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{2\pi}$$

$$= \frac{-1}{4\pi}[\cos 4\pi - 1]$$

$$= \frac{-1}{4\pi}[1 - 1]$$

$$\therefore (\phi_1,\phi_2) = 0$$

$$(\phi_1,\phi_1) = \int_0^{2\pi} \frac{\cos x}{\sqrt{\pi}} \cdot \frac{\cos x}{\sqrt{\pi}} dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \cos 2x dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{\pi}\left[\frac{x+\sin 2x}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}[2\pi + \sin 4\pi - 0]$$

$$= \frac{1}{2\pi}[2\pi + 0 - 0] = 1$$

$$\therefore (\phi_1,\phi_1) = 1$$

An orthonormal system of complex-valued functions or every interval of length 2π is given by

$\phi_n(x) = \frac{e^{i\pi x}}{\sqrt{2\pi}} = \frac{\cos \pi x + i\sin \pi x}{\sqrt{2\pi}}, \pi = 0,1,2,\ldots$

3.2. The Theorem on Best Approximation:

Theorem 1:

Let $\{\phi_0,\varphi_1,\phi_2,\ldots\}$ be orthonormal on I, and assume that $f \in L^2(I)$. Define two sequences of functions $\{s_n\}$ and $\{t_n\}$ on I as follows: $S_n(x) = \sum_{k=0}^{n} c_k\varphi_k(x), t_n(x) = \sum_{k=0}^{n} b_k\varphi_k(x)$

Where, $c_k = (f,\varphi_k)$ for $k = 0,1,2,\ldots$ - (1) and $b_0, b_1, b_2, \ldots$, are arbitrary complex numbers.

Then for each π we have $\|f - s_n\| \leqslant \|f - t_n\|$

Moreover, equality holds in (2) inf, $b_k = c_k$ for $k = 0,1,\ldots,n$

Proof:

Let $\{\phi_0,\phi_1,\phi_2,\ldots\}$ be an orthonormal on I

Define $s_\pi(x) = \sum_{k=0}^{n} c_k\phi_k(x)$ and $t_k(x) = \sum_{k=0}^{n} b_k\phi_k(x)$

where $c_k = (f, \phi_k)$ for $k = 0,1,2, \ldots$

$$\|f - t_n\|^2 = (f - t_n, f - t_n)$$
$$\|f - t_n\|^2 = (f, f) - (f, t_n) - (t_n, f) + (t_n, t_n) - - - - (1)$$

Now, $(f, f) = \| f \|^2 - - - - (2)$

$$(f, t_n) = \left(f, \sum_{k=0}^{n} b_k\varphi_k(x)\right)$$
$$= (f, b_0\varphi_0 + b_1\varphi_1 + \cdots + b_n\varphi_n)$$
$$= (f, b_0\varphi_0) + (f, b_1\varphi_1) + \cdots + (f, b_n\varphi_n)$$
$$= \bar{b}_0(f, \varphi_0) + \bar{b}_1(f, \varphi_1) + \cdots + \bar{b}_n(f, \varphi_n)$$
$$= \textstyle\sum_{k=0}^{n} \bar{b}_k(f, \varphi_k)$$

$\therefore (f, t_k) = \sum_{k=0}^{n} \bar{b}_k c_k$ .... (3)

$$(t_n, f) = (\overline{f, t_n})$$
$$= \overline{\textstyle\sum_{k=0}^{n} \bar{b}_k c_k} \; (\because \text{by equation } (3))$$
$$= \textstyle\sum_{k=0}^{n} b_k\bar{c}_k$$

$$\therefore (t_k, f) = \sum_{k=0}^{n} b_k\bar{c}_k \quad \ldots\ldots\ldots (4)$$

$$(t_n, t_n) = \left(\sum_{k=0}^{n} b_k\varphi_k, \sum_{k=0}^{n} b_k\varphi_k\right)$$
$$= (b_0\varphi_0 + b_1\varphi_1 + \cdots + b_n\varphi_n, b_0\varphi_0 + b_1\varphi_1 + \cdots + b_n\varphi_n)$$
$$= (b_0\varphi_0, b_0\varphi_0) + (b_0\varphi_0, b_1\varphi_1) + \cdots + (b_0\varphi_0, b_\pi\varphi_n)$$
$$+ \cdots + (b_n\varphi_n, b_0\varphi_0) + (b_n\varphi_n, b_1\varphi_1) + \cdots + (b_n\varphi_n, b_n\varphi_n)$$
$$= b_0\bar{b}_0(\varphi_0, \varphi_0) + \cdots + b_n\bar{b}_n(\varphi_0, \varphi_n) + .. + b_n\bar{b}_0(\varphi_n, \varphi_0) +$$
$$\cdots + b_n\bar{b}_n(\varphi_n, \varphi_\pi)$$
$$= b_0\bar{b}_0(\varphi_0, \varphi_0) + b_1\bar{b}_1(\varphi_1, \varphi_1) + \cdots + b_n\bar{b}_n(\varphi_n, \varphi_n)$$
$$= b_0\bar{b}_0 + b_1\bar{b}_1 + \cdots + b_n\bar{b}_n$$

$(t_n, t_n) = \sum_{k=0}^{M} b_k\bar{b}_k = \sum_{k=0}^{n} |b_k|^2$----(5)

Substitute (2), (3), (4) and (5) in (1)

$$\|f-t_k\|^2=\| f \|^2-\sum_{k=0}^{n} \bar{b}_k c_k-\sum_{k=0}^{n} b_k\bar{c}_k+\sum_{k=0}^{n} b_k\bar{b}_k$$

Add and Subtract, $\sum_{k=0}^{n} c_k\bar{c}_k$

$$\|f-t_n\|^2=\| f \|^2-\sum_{k=0}^{n} c_k\bar{c}_k+\sum_{k=0}^{n} c_k\bar{c}_k-\sum_{k=0}^{n} \bar{b}_k c_k$$

$$-\sum_{k=0}^{n} b_k\bar{c}_k+\sum_{k=0}^{n} b_k\bar{b}_k$$

$$=\| f \|^2-\sum_{k=0}^{n} c_k\bar{c}_k+\sum_{k=0}^{n} (b_k-c_k)\bar{b}_k$$

$$+\sum_{k=0}^{n} (b_k-c_k)(-\bar{c}_k)$$

$\|f-t_n\|^2=\| f \|^2-\sum_{k=0}^{n} c_k\bar{c}_k+\sum_{k=0}^{n} (b_k-c_k)\left(\bar{b}_k-\bar{c}_k\right)$ -----(6)

Similarly,

$$\|f-s_n\|^2=(f-s_n, f-s_n)$$

$$\|f-s_n\|^2=(f,f)-(f,s_n)-(s_n,f)+(s_n,s_n) \quad \ldots\ldots..(7)$$

$$(f,s_n)=\left(f,\sum_{k=0}^{n} c_k\phi_k(x)\right)=\sum_{k=0}^{n} \bar{c}_k c_k \quad \ldots\ldots\ldots\ldots(8)$$

$$(s_n,f)=\left(\bar{f},s_n\right)=\sum_{k=0}^{n} \overline{\bar{c}_k c_k}=\sum_{k=0}^{n} c_k\bar{c}_k \quad \ldots\ldots\ldots.(9)$$

$$(s_n,s_n)=\left(\sum_{k=0}^{n} c_k\phi_k,\sum_{k=0}^{n} c_k\phi_k\right)=\sum_{k=0}^{n} \underbrace{c_k}\,\bar{c}_k \quad \ldots\ldots\ldots..(10)$$

Substitute equation (8), (9) and (10) in (7)

$$\|f-s_n\|^2=\| f \|^2-\sum_{k=0}^{n} \bar{c}_k c_k-\sum_{k=0}^{n} c_k\bar{c}_k+\sum_{k=0}^{n} c_k\bar{c}_k$$

$$=\| f \|^2-\sum_{k=0}^{n} \bar{c}_k c_k \quad \ldots\ldots\ldots..(11)$$

Substitute equation (11) in (6) $\|f-t_n\|^2=\|f-s_n\|^2+\sum_{k=0}^{n} |b_k-c_k|^2$

$$\therefore \|f-t_k\| \geqslant \|f-s_n\| \quad \ldots\ldots\ldots.(12)$$

The RHS of (12) has its smallest value when $b_k = c_k$ for each k

Hence we have $\|f - s_n\| = \|f - t_n\|$

3.3. The Fourier Series of a Function Relative to an Orthonormal System:

Let $S = \{\phi_0, \phi_1, \phi_2, \ldots\}$ be orthonormal on I and assume that f $\in L^2(I)$. The notation,

$f(x) = \sum_{n=0}^{\infty} c_n \varphi_\pi(x)$ .... (1)

where $c_0, c_1, c_2, \ldots$ are given by the formulas

$$c_n = (f, \phi_n) = \int_I f(x)\overline{\phi_n(x)}dx \; (n = 0,1,2,\ldots)$$

The series in (1) is called the Fourier series of f relative to S and the number c_0, c_1, c_2 ... are called the Fourier coefficients of f relative to S.

Note:

When $I = [0,2\pi]$ and S is the system of trigonometric formations $\phi_0(x) = \frac{1}{\sqrt{2\pi}}$, $\phi_{2\pi-1}(x) = \frac{\cos \pi x}{\sqrt{\pi}}$ and $\phi_{2\pi}(x) = \frac{\sin \pi x}{\sqrt{\pi}} \cdot p_\pi(x) = \frac{e^{i\pi x}}{\sqrt{2\pi}} = \frac{\cos \pi x + i \sin \pi x}{\sqrt{2\pi}}$ the series is called the Fourier series generated by f. We write eqn(1) in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The coefficients bring given in the following formulas

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos \pi t \, dt,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin \pi t \, dt$$

in this case the integrals for a_n and b_n exist if $f \in L([0,2\pi])$.

3.4.Properties of the Fourier Coefficients:

Theorem 2:

Let $\{\varphi_0, \varphi_1, \varphi_2 \ldots\}$ be orthonormal on I, assume that $f \in L^2(I)$ and suppose that

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x)$$

Then

(a) The series $\sum |c_n|^2$ converges and satisfies the inequality $\sum_{n=0}^{\infty} |c_n|^2 \leqslant \| f \|^2$

(Bessel's inequality)

(b) The equation $\sum_{n=0}^{\infty}|c_m|^2 = \| f \|^2$ (Parseval's formula) holds if, we also have $\lim_{n\to\infty}\|f - s_n\| = 0$ where $\{s_n\}$ is the sequence of partial sums defined by

$s_n(x) = \sum_{k=0}^{n} c_k\varphi_k(x)$.

Note:

The Fourier Coefficients $c_n \to 0$ as $H \to \infty$. Since $\sum|c_n|^2$ converges. Then $\phi_n(x) = \frac{e^{i\pi x}}{\sqrt{2\pi}}$ and $I = [0,2\pi]$. We define $\lim_{n\to\infty}\int_0^{2\pi} f(x)e^{-inx}dx = 0$

$$\text{In other words} \Rightarrow \lim_{n\to\infty}\int_0^{2\pi} f(x)\cos\pi x dx = 0;$$

$$\lim_{n\to\infty}\int_0^{2\pi} f(x)\sin\pi x dx = 0;$$

Note:

$\| f \|^2 = |c_0|^2 + |c_1|^2 + \cdots$

which is equivalent to $\| x|^2 = |x_1|^2 + |x_0|^2 + \cdots + |x_n|^2$ for the length of the vector $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

3.5. The Riesz - Fischer Theorem:

Theorem 3

Assume $\{\phi_0, \phi_1, \ldots\}$ is a orthonormal on I. Let $\{c_n\}$ be any sequence of complex numbers such that $\sum|c_k|^2$ converges then there is a function f in $L^2(I)$ such that

$a)(f, \phi_k) = c_k$ for each $k \geqslant 0$.

b) $\| f \|^2 = \sum_{k=0}^{\infty}|c_k|^2$

Proof:

Given: $\{\varphi_0, \varphi_1, \ldots\}$ is orthonormal in I and $\sum|c_k|^2$ converges, let $S_n(x) = \sum_{k=0}^{n} c_k\varphi_k(x)$

(b) To prove: This is a function f in $L^2(I)$

such that $\lim_{n\to\infty}\|_f - s_n\| = 0$. Now, $\{S_n\}$ is a Cauchy Sequence in the semimetric space $L^2(I)$ because if $m > n$ we have

$$\|s_n - s_m\|^2 = \left\|\sum_{k=0}^{\pi} c_k\varphi_k - \sum_{k=0}^{m} c_k\varphi_k\right\|^2$$

$$= \| c_0\varphi_0 + c_1\varphi_1 + \cdots + c_n\varphi_n - c_0\varphi_0 - \cdots - c_n\varphi_n - c_{n+1}\varphi_{n+1} - c_m\varphi_m \|^2$$

$$= \|c_{n+1}\varphi_{n+1} + \cdots + c_m\varphi_m\|^2$$

$$= \left\| \sum_{k=k+1}^{m} c_k\phi_k \right\|^2 = \left(\sum_{k=n+1}^{m} c_k\phi_k, \sum_{k=n+1}^{m} c_k\phi_k \right)$$

$$= (c_{n+1}\dot{\phi}_{n+1} + c_{n+2}\dot{\phi}_{n+2} + \cdots + c_m\varphi_m, \\ c_{n+1}\varphi_{n+1} + c_{n+2}\varphi_{n+2} + \cdots + c_m\varphi_m)$$

$$= (c_{n+1}\varphi_{n+1}, c_{n+1}\varphi_{n+1}) + (c_{n+1}\varphi_{n+1}, c_{n+2}\varphi_{n+2} \\ \ldots + (c_{n+1}\varphi_{n+1}, c_m\varphi_m) + \cdots (c_{n+2}Q_{n+2}, c_{n+1}\varphi_{n+1}) + \\ (c_{n+2}Q_{n+2}, c_{n+2}\varphi_{n+2}) + \cdots + (c_{m+2}\varphi_{m+2}, c_m\varphi_m) + \\ +(c_m\varphi_m, c_{n+1}\varphi_{n+1}) + \cdots . +(c_m\overline{c_m}(\varphi_m, \varphi_m)$$

$$c_{n+1}\,\overline{c_{n+1}}(\phi_{n+1}, \phi_{n+1}) + \cdots + c_{n+2}, \overline{c_{n+2}}(Q_{n+2}, \phi_{n+2}) + \cdots + c_m\overline{c_m}(\phi_m, \phi_m)$$

$$\|s_n - s_m\|^2 = c_{n+1}\overline{c_{n+1}} + c_{n+2}\overline{c_{n+2}} + \cdots + c_m\overline{c_m}$$

$$= \textstyle\sum_{k=0}^{m} c_k\overline{c_k} = \sum_{k=0}^{m} \cdot |c_k|^2 < \varepsilon$$

$\therefore \|s_\pi - s_m\|^2 < \varepsilon$ $\{\because m$ and m are sufficiently large and $\sum|c_k|^2$converges$\}$

$\{S_n\}$ is a cauchy sequence in $L^2(I)$.

S_n converges to f

By theorem, (Let $\{f_n\}$ be a complex Value functions in $L^2(I)$.

Assume that for every $\varepsilon > 0$. There exists an integer N such that $\|f_n - f_m\| < \varepsilon$, whenever $\pi \geqslant \pi \geqslant N$, then there exists a function in $l^2(I)$

such that $\lim_{n\to\infty}\|f_n - f\| = 0$)

$\therefore$ There is a function f in $L^2(I)$ such that $\lim_{n\to\infty}\|s_n - f\| = 0$.

(a) We have to show that $(f, \varphi_k) = c_k$ for each $k \geqslant 0$

To prove: $(s_K, \varphi_K) = c_k$ if $n \geqslant k$

$$(s_k, \phi_k) = \left(\sum_{k=0}^{n} c_k\phi_k, \phi_k\right)$$
$$= \sum_{k=0}^{M} c_k(\phi_k, \phi_k)$$
$$= \sum_{k=0}^{M} c_k$$

$\therefore (s_k, \varphi_k) = c_k$ for $n \geqslant k$

$|(s_k, \varphi_k)| = |c_k|$

$|(s_k, \varphi_k) - (f, \varphi_k)| = |c_k - (f, \varphi_k)|.$

$= |c_k - (f, \varphi_k)| = |(s_k, \phi_k) - (f, \phi_k)|$

$\leq \|s_k - f\|$

$\lim_{n\to\infty} |c_k - (f, \phi_k)| \leq \lim_{k\to\infty} \|s_\pi - f\|$

$|c_k - (f, q_k)| = 0.$

$\|f - s_k\|^2 = (f, f) - (f, s_n) - (s_n, f) + (s_n, s_n)$

$$=\| f \|^2 - \sum_{k=0}^{\pi} |c_k|^2$$

$$\lim_{k\to\infty} \|f - s_k\|^2 = \lim_{k\to\infty} \left(\| f \|^2 - \sum_{k=0}^{n} |c_k|^2\right)$$

$$0 = \| f \|^2 - \sum_{k=0}^{\infty} |c_k|^2$$

$$\| f \|^2 = \sum_{k=0}^{\infty} |c_k|^2$$

Results:

There exists a Lebesgue Integral function whose Fourier series diverges every where

There exist continuous functions whose Fourier series diverge on an uncountable set

The Fourier series of a function in $L^2(I)$ converges almost everywhere on I

3.6. The Convergence and Representation Problems for Trigonometric Series:

Consider the trigonometric Fourier series generated by a function f which is Lebesgue-integrable on the interval I: $[0,2\pi]$, say $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

Two questions arise. Does the series converge at some point x in I? If it does converge at x, is its sum $f(x)$? The first question is called the convergence problem; the second, the representation problem. In general, the answer to both questions is "No." In fact, there exist Lebesgue-integrable functions whose Fourier series diverge everywhere, and there exist continuous functions whose Fourier series diverge on an uncountable set.

Ever since Fourier's time, an enormous literature has been published on these problems. The object of much of the research has been to find sufficient conditions to be satisfied by f in order that its Fourier series may converge, either throughout the interval or at particular points. We shall prove later that the convergence or divergence of the series at a particular point depends only on the behavior of the function in arbitrarily small neighborhoods of the point. (See Riemann's localization theorem.)

The efforts of Fourier and Dirichlet in the early nineteenth century, followed by the contributions of Riemann, Lipschitz, Heine, Cantor, Du Bois-Reymond, Dini, Jordan, and de la Vallée-Poussin in the latter part of the century, led to the discovery of sufficient conditions of a wide scope for establishing convergence of the series, either at particular points, or generally, throughout the interval.

After the discovery by Lebesgue, in 1902, of his general theory of measure and integration, the field of investigation was considerably widened and the names chiefly associated with the subject since then are those of Fejer, Hobson, W. H. Young, Hardy, and Little wood. Fejer showed, in 1903, that divergent Fourier series may be utilized by considering, instead of the sequence of partial sums $\{s_r\}$, the sequence of arithmetic means $\{\sigma_n\}$, where

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + \cdots + s_{n-1}(x)}{n}.$$

He established the remarkable theorem that the sequence $\{\sigma_n(x)\}$ is convergen: and its limit is $\frac{1}{2}[f(x+) + f(x-)]$ at every point in $[0,2\pi]$ where $f(x+)$ and $f(x-)$ exist, the only restriction on f being that it be Lebesgue-integrable on $[0,2\pi]$ (Theorem 3). Fejer also proved that every Fourier series, whether it converges or not, can be integrated term-by-term (Theorem 4) The most striking result on Fourier series proved in recent times is that of Lennart Carleson, a Swedish mathematician, who proved that the Fourier series of a function in $L^2(I)$ converges almost everywhere on I.

In this chapter we shall deduce some of the sufficient conditions for convergence of a Fourier series at a particular point. Then we shall prove Fejer's theorems. The discussion rests on two fundamental limit formulas which will be discussed first. These limit formulas, which are also used in the theory of Fourier integrals, deals with integrals depending on a real parameter α, and we are interested in the behavior of these integrals as $\alpha \to +\infty$. The first of these is a generalization of theorem 3 and is known as the Riemann-Lebesgue lemma.

3.7.The Riemann Lebesgue Lemma:

Theorem 4:

Assume that $f \in L(I)$. Then, for each real β ,we have

$$\lim_{\alpha\to+\infty} \int f(t)\sin(\alpha t+\beta)\,dt = 0 \quad \ldots\ldots\ (1)$$

Proof:

If f is a characteristic function of a compact interval $[a, b]$

$$\left|\int_a^b \sin(\alpha t+\beta)dt\right| = \left|\left[\frac{-\cos(\alpha t+\beta)}{\alpha}\right]_a^b\right|$$

$$= \left|\frac{-\cos(\alpha b+\beta)+(\cos\alpha a+\beta)}{\alpha}\right|$$

$$= \left|\frac{\cos(\alpha a+\beta)-\cos(\alpha b+\beta)}{\alpha}\right|$$

$$< \frac{|\cos(\alpha a+\beta)|+|\cos(\alpha b+\beta)|}{\alpha}$$

$$\leqslant \frac{2}{\alpha}$$

$$\lim_{\alpha\to+\infty}\int_a^b \sin(\alpha t+\beta)dt = 0.$$

The result is true if f is a constant on (a, b) and zero outside regardless of how we define $f(a)$ and $f(b)$.

To prove: For every Lebasque integral function f.

By the theorem, Assure that $f \in L(x)$ and Let $\varepsilon > 0$ be given. Then there exists a step function S and a function g in $L(I)$ such that $f = s + g$ where $\int_I |g| < \varepsilon$.

Assume that $f \in L(I)$

Given : $\varepsilon > 0$, f a step function s and $g \in L(I)$ such that $f = s + g$.

$$\left.\int_I |f - s| < \varepsilon/2\right\} \quad \dots\dots\dots (2)$$

The step function holds in (1), there is a positive M,

$\left|\int_I s(t)\sin(\alpha t + \beta)dt\right| < \varepsilon/2$, if $\alpha \geqslant M$ ………. (3)

If $\alpha \geqslant M$, we have,

$$\int_I f(t)\sin(\alpha t + \beta)dt = \int_I f(t)\sin(\alpha t + \beta)dt$$
$$- \int_I s(t)\sin(\alpha t + \beta)dt + \int_I s(t)\sin(\alpha t + \beta)dt$$
$$= \int_I (f(t) - s(t))\sin(\alpha t + \beta)dt + \int_I s(t)\sin(\alpha t + \beta)$$
$$\left|\int_I f(t)\sin(\alpha t + \beta)dt\right| \leq \int_I |f(t) - s(t)||\sin(\alpha t + \beta)|dt +$$
$$\left|\int_I s(t)\sin(\alpha t + \beta)\,dt\right| \qquad \leq \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon \text{ (by (2)ln(3))}$$
$$\therefore \left|\int_I f(t)\sin(\alpha t + \beta)dt\right| < \varepsilon.$$
$$\therefore \lim_{\alpha\to+\infty} \int_I f(t)\sin(\alpha t + \beta)dt = 0.$$

Note:

Take $\beta = 0$, We get

$$\lim_{\alpha\to+\infty} \int_{\mathrm{I}} f(t)\sin\alpha t dt = 0$$

Theorem 5:

If $f \in \mathrm{l}(-\infty, +\infty)$, we have

$$\lim_{\alpha\to+\infty} \int_{-\infty}^{\infty} f(t)\frac{1 - \cos\alpha t}{t}dt = \int_0^{\infty} \frac{f(t) - f(-t)}{t}dt, \text{ ------(1)}$$

whenever the Lebasque integral on the right exists.

Proof:

For each fixed x,

Consider $\frac{1-\cos xt}{t}$

$\lim_{t\to 0} \frac{1-\cos \alpha t}{t} = 0.$

The quotient $\frac{1-\cos \alpha t}{t}$ is continuous and bounded on $(-\infty, +\infty)$.

∴ The integral on the left of (1) exists as a lebesgue integral

$$\int_{-\infty}^{\infty} f(t)\frac{(1-\cos \alpha t)}{t}dt = \int_{-\infty}^{0} f(t)\cdot\frac{(1-\cos \alpha t)}{t}dt + \int_{0}^{\infty} f(t)\cdot\frac{(1-\cos \alpha t)}{t}dt$$

$$= \int_{-\infty}^{\infty} f(t)\frac{1-\cos \alpha t}{t}dt + \int_{0}^{\infty} f(-t)\frac{1-\cos \alpha t}{t}dt$$

$$- \int_{0}^{\infty} f(-t)\cdot\frac{1-\cos \alpha t}{t}dt + \int_{0}^{\infty} f(t)\frac{(1-\cos \alpha t)}{t}dt$$

$$= 0 + \int_{0}^{\infty} [f(t) - f(-t)]\frac{1-\cos \alpha t}{t}dt$$

$$= \int_{0}^{\infty} \frac{f(t)-f(-t)}{t}dt - \int_{0}^{\infty} \frac{f(t)-f(-t)}{t}\cos \alpha t dt$$

(i.e) $$\int_{-\infty}^{\infty} f(t)\frac{(1-\cos \alpha t)}{t}dt = \int_{0}^{\infty} \frac{f(t)-f(-t)}{t}dt - \int_{0}^{\infty} \frac{f(t)-f(-t)}{t}\cos t\ dt$$

$$\lim_{\alpha\to+\infty} \int_{-\infty}^{\infty} f(t)\frac{(1-\cos \alpha t)}{t}dt = \int_{0}^{\infty} \frac{f(t)-f(-t)}{t}dt$$

$$- \lim_{\alpha\to-\infty} \int^{\infty} \frac{f(t)-f(-t)}{t}\cos \alpha t dt$$

$$= \int_{0}^{\infty} \frac{f(t)-f(-t)}{t}dt$$

$$\therefore \lim_{\alpha\to+\infty} \int_{-\infty}^{\infty} f(t)\frac{(1-\cos xt)}{t}dt = \int_{0}^{\infty} \frac{f(t)-f(-t)}{t}dt$$

3.8.Dirichlet Integrals:

Integrals of the form $\int_0^{\delta} g(t)\frac{\sin \alpha t}{t}dt$ (called Dirichlet Integrals)

[Bonnet's theorem:

Let g be continuous and assume that $f \nearrow$ on $[a, b]$. Let A and B be two real numbers satisfying the condition $A \leqslant f(a+)$ $B \geqslant f(b-)$. Then there exists a point x_0 in $[a, b]$ such that

(i) $$\int_a^b f(x)g(x)dx = A\int_a^{x_0} g(x)dx + B\int_{x_0}^b g(x)dx$$

In particular if $f(x) \geqslant 0$ for all $x \in [a,b]$ such that

(ii) $\int_a^b f(x)g(x)dx = B\int_x^b g(x)dx$ where $x_0 \in [a,b]$]

Theorem 6: (Jordan)

If g is of bounded variation on $[0,\delta]$. Then $\lim_{\alpha\to+\infty} \frac{2}{\pi}\int_0^\delta g(t)\frac{\sin \alpha t}{t}dt = g(0+)$

Proof:

To prove: $\lim_{\alpha\to+\infty} \frac{2}{\pi}\int_0^\delta g(t)\frac{\sin \alpha t}{t}dt = g(0+)$

It is enough to prove that the case in which g is increasing of $[0,\delta]$

If $\alpha > 0$ and if $a < h < \delta$, we have

$$\int_0^\delta g(t)\frac{\sin \alpha t}{t}dt = \int_0^k g(t)\frac{\sin \alpha t}{t}dt + \int_h^\delta g(t)\frac{\sin xt}{t}dt$$

$$= \int_0^h g(t)\frac{\sin \alpha t}{t}dt - \int_0^h g(0+)\frac{\sin \alpha t}{t}dt$$

$$+ \int_0^h g(0+)\frac{\sin \alpha t}{t}dt + \int_h^\delta g(t)\frac{\sin \alpha t}{t}dt$$

$$\int_0^h \frac{\sin \alpha t}{t}dt = \int_0^h [g(t) - g(0+)]\frac{\sin \alpha t}{t}dt$$

$$+g(0+)\int_0^h \frac{\sin \alpha t}{t}dt + \int_h^h g(t)\frac{\sin \alpha t}{t}dt\Big)$$

$$\int_0^h g(t)\frac{\sin \alpha t}{t}dt = I_1(\alpha,h) + I_2(\alpha,h) + I_3(\alpha,h) - - - (1)$$

Now, $I_1(\alpha,h) = \int_0^h [g(t) - g(0+)]\frac{\sin \alpha t}{t}dt$

$$= [g(h) - g(0+)]\int_C^h \frac{\sin \alpha t}{t}dt$$

$$|I_1(\alpha,h)| = |g(h) - g(0+)|\left|\int_e^h \frac{\sin \alpha t}{t}dt\right|$$

choose $M > 0$ so that $\left|\int_a^b \frac{\sin \alpha t}{t}dt\right| < M \forall b \geqslant a \geqslant 0$

It follows that $\left|\int_a^b \frac{\sin \alpha t}{t}dt\right| < M$ for every $b \geqslant a \geqslant 0$ if $\alpha > 0$

Let $\varepsilon > 0$ be given and choose h in $(0,\delta)$

So that $|g(n) - g(0+)| < \frac{\varepsilon}{3M}$ since $g(t) - g(0+) \geqslant 0$

if $0 \leqslant t \leqslant h$

$$|I_1(\alpha,h)| < \frac{\varepsilon}{3M} \times M = \frac{\varepsilon}{3}$$

i.e, $|I_1(\alpha,h)| < \frac{\varepsilon}{3}$. -------(2)

Now, $I_2(\alpha,h) = g(0+)\int_0^h \frac{\sin \alpha t}{t} dt$

Put $y = \alpha t \quad dy = \alpha dt$

when $t = 0 \Rightarrow y = 0, \quad t = h \Rightarrow y = \alpha h$

$$\int_0^h \frac{\sin \alpha t}{t} dt = \int_0^{\alpha h} \frac{\sin y}{\frac{y}{\alpha}} \cdot \frac{dy}{\alpha}$$

$$= \int_0^{\alpha h} \frac{\sin y}{y} \cdot dy = \int_0^{\alpha h} \frac{\sin t}{t} dt$$

$$I_2(\alpha,h) = g(0+) \int_0^{\alpha h} \frac{\sin t}{t} dt$$

$$\therefore I_2(\alpha,h) = \frac{\pi}{2}\, g(0+) \text{ as } \alpha \to +\infty \quad \ldots\ldots\ldots(3)$$

$$\left(\because \int_0^{\alpha h} \frac{\sin t}{t} dt \to \pi/2 \text{ as } \alpha \to +4\right)$$

Now, $I_3(\alpha,h) = \int_h^\delta g(t) \frac{\sin \alpha t}{t} dt$

Apply Riemann-lebesque lemma to $I_3(\alpha,h)$ (since the integral $\int_h^s \frac{g(t)}{t} dt$ exists)

$\therefore I_3(x,h) \to 0$ as $x \to +\infty$. For the same h we car choose A so that $\alpha \geqslant A$ implies that

$|I_3(\alpha,h)| \varepsilon/3$---(4)

(3) $\Rightarrow |I_2(x,h) - \pi/2 g(0+)| < \varepsilon/3 - (5)$

Then for $\alpha \geqslant A$, combine (2), (4) and (5) We have

$$|\, I_1(\alpha,h) + I_2(\alpha,h) - \pi/2 g(0+) + I_3(\alpha,h)\,| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

i. , $$\left|\int_0^\delta g(t) \frac{\sin \alpha t}{t} dt - \pi/2 g(0t)\right| < \varepsilon$$

$$\int_0^\delta g(t) \frac{\sin \alpha t}{t} dt = \pi/2 g(0+)$$

$$\therefore \lim_{\alpha \to +\infty} 2/\pi \int_0^\delta g(t) \frac{\sin \alpha t}{t} dt = g(0+).$$

Theorem 7:

Assume that $g(0+)$ exists and suppose that for some $\delta > 0$ the Lebesgue integral $\int_0^\delta \frac{g(t)-g(0t)}{t} dt$ exists. They we have $\lim_{\alpha\to+\infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin \alpha t}{t} dt = g(0+)$

Proof:

Given: $g(0+)$ exists and $\int_0^\delta \frac{g(t)-g(0t)}{t} dt$ exists

$$\int_0^\delta g(t) \cdot \frac{\sin \alpha t}{t} dt = \int_0^\delta g(t) \frac{\sin \alpha t}{t} dt - g(0+) \int_0^\delta \frac{\sin \alpha t}{t} dt$$

$$+g(0+) \int_0^s \frac{\sin xt}{t} dt$$

$$\int_0^\delta g(t) \cdot \frac{\sin \alpha t}{t} dt = \int_0^s \frac{g(t) - g(0+)}{t} \sin \alpha t dt + g(0t) \int_0^\delta \frac{\sin \alpha t}{t} dt$$

$$= \int_0^\delta \frac{g(t) - g(0+)}{t} \sin \alpha t dt + g(0t) \int_0^\delta \frac{\sin t}{t} dt$$

When $\alpha \to +\infty$ first term on the RHS is zero by using Riethan-Lebesgue and the second term approaches $\pi/2$

$$\lim_{\alpha\to+\infty} \int_0^S \frac{g(t)\sin \alpha t}{t} dt = \pi/2 g(0+)$$

$$\Rightarrow \lim_{x\to+\infty} 2/\pi \int_0^\delta \frac{g(t) \sin \alpha t}{t} dt = g(0+)$$

3.9. An Integral Representation for the Partial Sums of a Fourier series:

A function f is said to be periodic with period $p \neq 0$ if f is defined on R and if $f(x+p) = f(x) \forall x$. The partial sums of a Fourier series in terms of the function

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \begin{cases} \dfrac{\sin(\pi + 1/2)t}{2\sin t/2} & \text{if} \quad t \neq 2m\pi \\ & \quad \text{(where } m \text{ is ar integer)} \\ n + 1/2 & \text{if} \quad t = 2m\pi \end{cases}$$

(m is an integer)

The function D_n is called the Dirichlet's kernel.

Theorem 8:

Assume that $f \in L([0,2\pi])$ and suppose f is periodic with period 2π. Let $\{S_\pi\}$ denote the sequence of partial sum of the Fourier series generated by f, say,

$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), n = 1,2, \dots.$ Then we have the integral representation

$$S_n(x) = \frac{2}{\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2} D_n(t)dt.$$

Proof:

The Fourier Coefficient of f are given by the integral

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos ntdt, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin tdt$$

$$S_n(x) = \frac{1}{\pi}\int_0^\pi \mathrm{f(t)} + \sum_{k=1}^{n} [\ + \frac{1}{\pi}\int_0^{2\pi} f(t)cosktcoskxdt + \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt \sin kxdt$$

$$S_n(x) = \frac{1}{\pi}\int_0^{2\pi} f(t)\left\{\frac{1}{2} + \sum_{k=1}^{\pi} [\cos k + \cos kx + \sin kt \sin kx]dt\right.$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(t)\left\{\frac{1}{2} + \sum_{k=1}^{\pi} \cos k(t-x)\right\}dt$$

$$S_n(x) = \frac{1}{\pi}\int_0^{2\pi} f(t)D_n(t-x)dt$$

since both f and D_n are periodic with period 2π, we can replace the interval of integration by $[x - \pi, x + \pi]$.

$$s_n(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(t)D_n(t-x)dt$$

Put $u = t - x \Rightarrow t = u + x.$

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(u+x)D_n(u)dx$$

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{0} f(u+x)D_n(u)du + \frac{1}{\pi}\int_0^\pi f(u+x)D_n(u)du\Big)$$

For 1[st] text

$u = -t$
$du = -dt$

u	$-\pi$	0
$-t$	π	0

For $2^{\pi d}$ term $u = t$

du = dt

$$S_\pi(x) = \frac{1}{\pi}\int_\pi^0 f(x-t)D_n(-t)(-dt) + \frac{1}{\pi}\int_0^\pi f(x+t)D_n(t)dt$$

$$= \frac{1}{\pi}\int_0^\pi f(x-t)D_n(t)dt + \frac{1}{\pi}\int_0^\pi f(x+t)D_n(t)dt$$

$$= \frac{1}{\pi}\int_0^\pi [f(x-t)+f(x+t)]D_n(t)\mathrm{dt}$$

$$S_\pi(x) = \frac{2}{\pi}\int_0^\pi \left[\frac{f(x-t)+f(x+t)]}{2}D_n(t)dt\right.$$

3.10. Riemann's Localization Theorem:

Theorem 9:

Assume that $f \in L([0,2\pi])$ and suppose f has period 2π. Then the fourier series generated by f will converge for a given value of x if and only if for some positive $s < \pi$, the following limit exists $\lim_{\pi\to\infty} \frac{2}{\pi}\int_0^s \frac{f(x+t)+f(x-t)}{2} \cdot \frac{\sin(n+1/2)t}{t} dt$ in which case the value of this limit is the sum of the Fourier series.

Proof.

$$\text{let } \sin(x) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos kx + b_k \sin kx] \ldots (1)$$

Integral representation of the partial sums of the Fourier series is

$$S_n(x) = \frac{2}{\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2} D_\pi(t)dt$$

from (2), the Fourier series generated by If will converge at a point x iff the following limit exists,

$$\lim_{n\to\infty} \frac{2}{\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2} \cdot \frac{\sin(n+1/2)t}{2\sin 1/2t} dt$$

in which case the value of this limit will be the sum of the series

Replace $t = 2\sin\frac{1}{2}t$ in (3)

Since, Riemann - Lebesgue Lemma allows this replacement without affecting the existence of the value of the limit.

i,e , $\lim_{n\to\infty} \frac{2}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} \cdot \frac{\sin(n+1/2)t}{tL^2(4)} dt$

(4) $-(3)$

$$\lim_{n\to\infty} \frac{2}{\pi}\int_0^{\pi} \left[\frac{1}{t} - \frac{1}{2\sin 1/2t}\right] \frac{f(x+t)+f(x-t)}{2} \cdot \sin\left(n+\frac{1}{2}\right) t dt = 0$$

Because the function F defined by the equation

$$F(t) = \begin{cases} \frac{1}{t} - \frac{1}{2\sin\frac{1}{2}t}, & \text{if } 0 < t \leqslant \pi \\ 0, & \text{if } t = 0 \end{cases}$$

is continuous on $[0, \pi]$. (ie) F is continuous.

Assume that $f(x) = 1$

Then, $a_0 = 2, a_k = 0 = b_k \ (k \geqslant 1)$

substitute the Value in (1),

$\Rightarrow \ S_n(x) = 1$

From (2), we have

$$S_n(x) = \frac{2}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} D_n(t) dt \qquad (5)$$

$$t = \frac{2}{\pi}\int_0^{\pi} D_n(t) dt$$

For any arbitrary $f \in L([0,2\pi])$

$$f(x) = \frac{2}{\pi}\int_0^{\pi} f(x) D_n(t) dt$$

Equation (5) - (6)

$$S_\pi(x) - f(x) = -f(x)\frac{2}{\pi}\int_0^\pi \frac{\left[\frac{f(x+t)+f(x-t)}{2}\right.}{2} - f(x)]D_{n(t)}dt$$

$$\lim_{n\to\infty}[S_n(x) - f(x)] = \lim_{k\to\infty}\frac{2}{\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2} - f(x) = 0$$

$$\Rightarrow \lim_{n\to\infty}[S_n(x) - f(x)] = 0.$$

∴ The convergence problem for the Fouriers series used for finding conditions on f which will gaurentee the existence of the following limit

$$\lim_{n\to\infty}\frac{2}{\pi}\int_0^{8\pi}\frac{f(x+t)+f(x-t)}{2}\cdot\frac{\sin(n+1/2)t}{t}dt$$

Then fox 'any $\delta < \pi$, we have.

$$\lim_{n\to\infty}\frac{2}{\pi}\int_0^{\delta}\frac{f(x+t)+f(x-t)}{2}\cdot\frac{\sin(n+1/2)t}{t}dt \text{ exist}$$

3.11. Sufficient Condition for the Convergence of a Fourier Series at a particular point:

Assume that $f \in ([0,2\pi])$ and suppose that f has period 2 π, *consider a fixed x in [0,2 π]*

and a positive $s < \pi$. Let
$$g(t) = \frac{f(x+t)+f(x-t)}{2} \text{ if } t \in [0,\delta]$$
$$s(x) = g(0+) = \lim_{t\to 0^+}\frac{f(x+t)+f(x-t)}{2}$$

whenever this limit exists. Note that $s(x) = f(x)$ if f is continuous at x.

3.12. Cesare Summability of Fourier Series:

Theorem 10:

Assume that $f \in L([0,2\pi])$ and suppose that f is periodic with period 2π. Let S_n denote the n^{th} partial sum of the Fourier series generated by f and

$$\sigma_\pi(x) = \frac{S_0(x) + S_1(x) + \cdots + S_{n-1}(x)}{n}\,(n = 1,2,\dots)$$

Then we have the integral representation $\sigma_\pi(x) = \frac{1}{n\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2}\frac{\sin^2\frac{1}{2}\pi t}{\sin^2\frac{1}{2}t}dt$

Proof:

Let S_n denote the n^{th} partial sum of the Fourier series generated by

$$\sigma_\pi(x) = \frac{s_0(x) + s_1(x) + \cdots + s_{\pi-1}(x)}{n}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} s_k(x)$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} \frac{2}{\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2} D_k(t)dt$$

$$\sigma_n(x) = \frac{1}{\pi}\int_0^\pi [f(x+t)+f(x-t)]k_n(t)dt$$

$$k_n(t) = \frac{1}{2n\sin t/2}\sum_{j=1}^{n} \sin(j-1/2)t$$

$$= \frac{1}{2n\sin t/2}\sum_{j=1}^{n} \sin(2j-1)t/2$$

$$k_n(t) = \frac{1}{2n\sin t/2}\cdot\frac{\sin^2\frac{nt}{2}}{\sin t/2}$$

Substitute Equation (2) in (1)

$$\sigma_n(x) = \frac{1}{n\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2}\cdot\frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt$$

Note:

If $f = 1$, then $s_0(x) = s_1(x) = s_2(x) = \cdots = s_{m-1}(x) = 1$

$$\therefore \sigma_n(x) = \frac{1+1+\cdots 1}{n}(n \text{ times }) = \frac{n}{n} = 1$$

$$\Rightarrow \sigma_n(x) = S_n(x) \text{ for each } n$$

Hence, $\frac{1}{n\pi}\int_0^\pi \frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt = 1$

$\because$ For given any number s, we get

$$\sigma_n(x) - s = \frac{1}{n\pi}\int_0^\pi \left[\frac{f(x+t)+f(x-t)}{2} - s\right]\frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt$$

If we can choose a value for such that the integral on the right of (1) tends to 0 as $n \to \infty$

$\therefore \sigma_n(x) \to s$ as $n \to \infty$

Theorem 11: (Fejer)

Assume that $f \in L([0,2\pi])$ and suppose f is periodic with period 2π. Define a function by the following equation: $s(x) = \lim_{t\to 0+} \frac{f(x+t)+f(x-t)}{2}$ whenever the limit exists. Then, for each x for which $s(x)$ is defined, the fourier series generated by f is cesaro summable and has $(c,1)$ sum $s(x)$. (i.e.,) we have $\lim_{n\to\infty} \sigma_n(x) = s(x)$ where $\{\sigma_n\}$ is the sequence of arithmetic means defined by $\sigma_n(x) = \frac{S_0(x)+S_1(x)+\cdots+S_{n-1}(x)}{n}, n = 1,2,\ldots$ If, in addition, f is continuous on $[0,2\pi]$, then the sequence (σ_n) converges uniformly to f on $[0,2\pi]$.

Proof:

Let $g_x(t) = \frac{f(x+t)+f(x-t)}{2} - s(x)$

Wherever $S(x) = \lim_{t\to 0^+} \frac{f(x+t)+f(x-t)}{2}$

$$g_x(t) = \frac{f(x+t)+f(x-t)-2\ s(x)}{2}$$

Then $gx(t) \to 0$ as $t \to 0^+$

$\therefore$ For given $\varepsilon > 0, \exists$ a positive $\delta < \pi$ such that $|g_x(t)| < \varepsilon/2$ whenever $0 < t < \delta$.

theorem 10,

$$\sigma_n(x) - s(x) = \frac{1}{n\pi}\int_0^{\pi} \left[\frac{f(x+t)+f(x-t)}{2} - sx^2\right] \frac{\sin^2 \frac{nt}{2}}{\sin^2 t/2} dt$$

Divide the interval of integration in to two subintervals $[0,\delta]$ and $[\delta,\pi]$

$$\sigma_n(x) - s(x) = \frac{1}{n\pi}\int_0^{s} \left[\frac{f(x+t)+f(x-t)-2s(x)}{2}\right] \frac{\sin^2 nt/2}{\sin^2 t/2} dt$$

$$+\frac{1}{n\pi}\int_\delta^{\pi} \left[\frac{f(x+t)+f(x-t)-2s(x)}{2}\right] \frac{\sin^2 nt/2dt}{\sin^2 t/2}$$

$$\sigma_n(x) - s(x) = \frac{1}{n\pi}\int_0^{\delta} g_x(t) \frac{\sin^2 nt/2}{\sin^2 t/2} dt + \frac{1}{n\pi}\int_\delta^{\pi} gx(t) \frac{\sin^2 nt/2d}{\sin^2 t/2}$$

$$|\sigma_n(x) - s(x)| \le \left|\frac{1}{n\pi}\int_0^{s} g(t) \cdot \frac{\sin^2 \frac{nt}{2}}{\frac{\sin^2 t}{2}} dt\right| + \left|\frac{1}{n\pi}\int_\delta^{\pi} g_x(t) \frac{\sin^2 \frac{nt}{2}}{\frac{\sin^2 t}{2}} dt\right| \ on[0,s]$$

$$\left|\frac{1}{n\pi}\int_0^s g_x(t)\frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt\right| = \frac{1}{n\pi}\int_0^S |g_x(t)|\frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt$$

$$< \frac{\varepsilon}{2n\pi}\int_0^\delta \frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt$$

$$< \frac{\varepsilon}{2}\left(\because \frac{1}{n\pi}\int_0^s \frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt = 1\right) on\ [\delta,\pi]$$

$$\left|\frac{1}{n\pi}\int_\delta^\pi g_x(t)\frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt\right| = \frac{1}{n\pi}\int_\delta^\pi |g_x(t)|\frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt$$

$$\leqslant \frac{1}{n\pi\sin^2\delta/2}\int_{\frac{\pi}{8}}^\pi |g_x(t)|dt$$

$$(\because \text{ for } t \geqslant \delta, D_n(t) \leqslant \frac{1}{\sin^2\delta/2})$$

$$\leqslant \frac{I(x)}{n\pi\sin^2\delta/2} \text{ where, } I(x) = \int_0^\pi |g_x(t)|dt$$

Now, choose N so that $\frac{I(x)}{N\pi\sin^2\delta/2} < \varepsilon/2$

Then for $n \geqslant N$,

From equation (2) $\Rightarrow$

$|\sigma_n(x) - s(x)| < \varepsilon.$

$\therefore \sigma_n(x) \to s(x)$ as $n \to \infty$

If f is continuous on $[0,2\pi]$, then by periodicity, f is bounded on R and there is an M such that $|g_x(t)| \leqslant M\ \forall x$.

Replace $I(x)$ by πM

$$(k) \Rightarrow \left|\frac{1}{n\pi}\int_\delta^\pi g_x(t)\frac{\sin^2\frac{nt}{2}}{\sin^2 t/2}dt\right| \leqslant \frac{\pi M}{n\pi\sin^2\delta/2}$$

choose $\frac{\pi M}{N\pi \sin^2 \delta/2} < \varepsilon/2$ for $n \geqslant N$.

3.13. Consequences of Fejer's Theorem:

Theorem 12:

Let f be contimuous on $[0,2\pi]$ and periedic with period 2π. Let $\{s_n\}$ denote the sequence of partial sums of the Fourier series generated by f, say

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$. .... (1)

Then we have:

a) $\lim_{n\to\infty} s_n = f \ on \ [0,2\pi]$

b) $\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ (Parseval's formula).

c) The Fourier series can be integrated term by term. That is, for all x we hate

$$\int_0^x f(t)dt = \frac{a_0 x}{2} + \sum_{n=1}^{\infty} \int_0^x (a_n \cos nt + b_n \sin nt)dt.$$

the integrated series being uniformly convergent on every interval, even if the Fourier series in equation (1) diverges.

d) If the Fourier series in equation (1) converges for some x, then it concerges on $f(x)$.

Proof:

Applying formula (3) of Theorem 1, with $f_n(x) = \sigma_n(x) = (1/n)\sum_{n=1}^{n-1} s_1(x)$.

we obtain the inequality

$\int_0^{2\pi} |f(x) - s_n(x)|^2 dx \leq \int_0^{2\pi} |f(x) - \sigma_n(x)|^2 dx$. .... (2)

But, since $\sigma_n \to f$ uniformly on $[0,2\pi]$. it follows that $\lim_{n\to\infty} \sigma_n = f$ on $[0,2\pi]$. And (2) implies (a).

Part (b) follows from (a) because of Theorem 2. Part (c): also follows from (a). Finally, if $\{s_n(x)\}$ converges for then $\{\sigma_n(x)\}$ must converge to the same limit. But since $\sigma_n(x) \to f(x)$ that $s_n(x) \to f(x)$, which proves (d).

3.14. The Weierstrass Approximation Theorem:

Fejer's theorem can also be used to prove a famous theorem of Weierstrass which states that every continuous function on a compact interval can be uniformly approximated by a polynomial. More precisely, we have:

Theorem 13:

Let f be real-valued and continuous on a compact interval $[a, b]$ Then for every $\varepsilon > 0$. there is a polynomial p (which may depend on c) such that

$|f(x) - p(x)| < \varepsilon$ for every x in $[a, b]$. …….. (1)

Proof:

If $t \in [0, \pi)$, let $g(t) = f[a + t(b - a)/\pi]$;

If $t \in [\pi, 2\pi]$, lat $g(t) = f[a + (2\pi - l)(b - a)/\pi]$ and define g outside $[0,2\pi]$ so that g has period 2 . For the ε given in the theorem, we can apply Fejer's theorem to find a function defined by an equation of the form

$$\sigma(t) = A_0 + \sum_{k=1}^{N} (A_k \cos kt + B_A \sin kt)$$

such that $|g(t) - \sigma(t)| < \varepsilon/2$ for every t in $[0,2\pi]$. (Note that N, and hence 0 , depends on c.) Since σ is a finite sum of trigonometric functions, it generates a power series expansion about the origin which converges uniformly on every finite interval. The partial sums of this power series expansion constitute a sequence o! polynomials, say $\{p_n\}$, such that $p_n \to \sigma$ uniformly on $[0.2\pi]$. Hence, for the same ε, there exists an m such that

$$|p_m(t) - \sigma(t)| < \frac{\varepsilon}{2}, \text{ for every } t \text{ in } [0,2\pi].$$

Therefore we have $|p_m(t) - g(t)| < c$, for every t in $[0,2\pi]$. …… (2)

Now define the polynomial p by the formula $p(x) = p_m[\pi(x - a)/(b - a)]$. Then inequality (2) becomes (1) when we put $t = \pi(x - a)/(b - a)$.

Unit IV

Multivariable Differential Calculus - Introduction - The Directional derivative - Directional derivative and continuity - The total derivative - The total derivative expressed in terms of partial derivatives - The matrix of linear function - The Jacobian matrix - The chain rule - Matrix form of chain rule - The mean - value theorem for differentiable functions - A sufficient condition for differentiability - A sufficient condition for equality of mixed partial derivatives - Taylor's theorem for functions of R^n to R^1

Chapter 4: Section 4.1 to 4.14

Multivariable Differential Calculus

4.1 Introduction:

Partial derivatives of functions from R^n to R^1 were discussed briefly in Chapter 5. We also introduced derivatives of vector-valued functions from R^1 to R^*. This chapter extends derivative theory to functions from R^* to $R^{\mathbf{m}}$.

The partial derivative is a somewhat unsatisfactory generalization of the usual derivative because existence of all the partial derivatives $D_1 f, \ldots, D_m f$ at a particular point does not necessarily imply continuity of f at that point. The trouble with partial derivatives is that they treat a function of several variables as a function of one variable at a time. The partial derivative describes the rate of change of a function in the direction of each coordinate axis. There is a slight generalization, called the directional derivative, which studies the rate of change of a function in an arbitrary direction. It applies to both real- and vector-valued functions.

4.2 The Directional Derivative:

Let S be a subset of R^n, and let $f: S \to R''$ be a function defined on S with values in $R^{\boldsymbol{m}}$. We wish to study how f changes as we move from a point $\mathbf{c}$ in S along a line segment to a nearby point $\mathbf{c} +$ u, where u $\neq 0$. Each point on the segment can be expressed as c $+ h$a, where h is real. The vector u describes the direction of the line segment. We assume that c is an interior point of S. Then there is an n-ball $B(\mathbf{c}; r)$ lying in S, and, if h is small enough, the line segment joining c to $c + hu$ will lie in $B(c; r)$ and hence in S.

Definition 1:

The directional derivative of f at c in the direction n, denoted by the symbol $f'(c;u)$, is defined by the equation $f'(c;x) = \lim_{h\to 0} \frac{f(c+hu)-f(c)}{h}$ ....(1)

whenever the limit on the right exists.

Note: Some authors require that $\| a \| = 1$, but this is not assumed here.

Examples:

1. The definition in (1) is meaningful if $u = 0$. In this case $f'(c;0)$ exists and equals 0 for every c in S.
2. If $u = u_k$, the k th unit coordinate vector, then $f'(c;u_k)$ is called a partial derivative and is denoted by $D_k f(c)$.
3. If $f = (f_1, \dots, f_m)$, then $f'(c;u)$ exists if and only if $f_k'(c;u)$ exists for each $k = 1,2,\dots,m$, in which case $f'(c;u) = (f_1'(c;u), \dots, f_m'(c;u))$ In particular, when $v = a_k$ we find $D_k f(c) = (D_k f_1(c), \dots, D_k f_m(c))$ .... (2)
4. If $F(t) = f(c+tu)$, then $F'(0) = f'(c;u)$. More generally, $F'(t) = f'(c+tu;)$ if either derivative exists.
5. If $f(x) = \| x \|^2$, then $F(t) = f(c+tu) = (c+tu)\cdot(c+tu)$ $= \| c \|^2 + 2tc\cdot u + t^2 \| u \|^2$

   so $F'(t) = 2c\cdot u + 2t \| u \|^2$; hence $F'(0) = f'(c;u) = 2c\cdot u$.
6. Linear functions. A function $f: R^n \to R^m$ is called linear if $f(ax+by) = af(x) + bf(y)$ for every x and y in R^n and every pair of scalars a and b. If f is linear, the quotient on the right of (1) simplifies to $f(u)$, so $f'(c;u) = f(u)$ for every c and every u.

4.3. Directional Derivatives and Continuity:

If $f'(c;u)$ exists in every direction u, then in particular all the partial derivatives $D, f(c), \dots, D_n f(c)$ exist. However, the converse is not true. For example, consider the real-valued function

$$f: R^2 \to R^1 \text{ given by } f(x,y) = \begin{cases} x+y & \text{if } x = 0 \text{ or } y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then $D_1 f(0,0) = D_2 f(0,0) = 1$. Nevertheless, if we consider any other direction $u = (a_1, a_2)$, where $a_1 \neq 0$ and $a_2 \neq 0$, then $\frac{f(0+hu)-f(0)}{h} = \frac{f(hu)}{h} = \frac{1}{h}$, and this does not tend to a limit as $h \to 0$.

A rather surprising fact is that a function can have a finite directional derivative $f'(c; u)$ for every u but may fail to be continuous at c. For example, let $f(x, y) = \begin{cases} xy^2/(x^2 + y^4) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

Let $u = (a_1, a_2)$ be any vector in R^2. Then we have $f(0 + hu) - f(0) = \frac{f(ha_1, ha_2)}{h} = \frac{a_1 a_2^2}{a_1^2 + h^2 a_2^4}$,

and hence $f'(0; u) = \begin{cases} a_2^2/a_1 & \text{if } a_1 \neq 0 \\ 0 & \text{if } a_1 = 0 \end{cases}$

Thus, $f'(0; u)$ exists for all u. On the other hand, the function f takes the value $\frac{1}{2}$ at each point of the parabola $x = y^2$ (except at the origin), so f is not continuous at (0,0), since $f(0,0) = 0$.

Thus we see that even the existence of all directional derivatives at a point fails to imply continuity at that point. For this reason, directional derivatives, like partial derivatives, are a somewhat unsatisfactory extension of the one-dimensional concept of derivative. We turn now to a more suitable generalization which implies continuity and, at the same time, extends the principal theorems of one-dimensional derivative theory to functions of several variables. This is called the total derivative.

4.4 The Total Derivative:

In the one-dimensional case, a function f with a derivative at c can be approximated near c by a linear polynomial. In fact, if $f'(c)$ exists, let $E_c(h)$ denote the difference

$E_c(h) = \frac{f(c+h) - f(c)}{h} - f'(c)$ if $h \neq 0$, ... (3)

and let $E_c(0) = 0$. Then we have $f(c + h) = f(c) + f'(c)h + hE_c(h)$ .... (4)

an equation which holds also for $h = 0$. This is called the first-order Taylor formula for approximating $f(c + h) - f(c)$ by $f'(c)h$. The error committed is $hE_c(h)$. From (3) we see that $E_c(h) \to 0$ as $h \to 0$. The error $hE_c(h)$ is said to be of smaller order than h as $h \to 0$.

We focus attention on two properties of formula (4). First, the quantity $f'(c)h$ is a linear function of h. That is, if we write $T_c(h) = f'(c)h$, then $T_c(ah_1 + bh_2) = aT_c(h_1) + bT_c(h_2)$.

Second, the error term $hE_c(h)$ is of smaller order than h as $h \to 0$. The total derivative of a function f from R^n to R^m will now be defined in such a way that it preserves these two properties.

Let $f: S \to R^m$ be a function defined on a set S in R^n with values in R^m. Let c be an interior point of S, and let $B(c; r)$ be an n-ball lying in S. Let v be a point in R^n with $\| v \| < r$, so that $c + v \in B(c; r)$.

Definition 2:

The function f is said to be differentiable at c if there exists a linear function $T_c: R^n \to R^m$ such that $f(c+v) = f(c) + T_c(v) + \| v \| E_e(v)$ ....(5) where $E_c(v) \to 0$ as $v \to 0$.

Note:

Equation (5) is called a first-order Taylor formula. It is to hold for all v in R^* with $\| v \| < r$. The linear function T_c is called the total derivative of f at c. We also write (5) in the form

$f(c+v) = f(c) + T_c(v) + o(\| v \|)$ as $v \to 0$.

The next theorem shows that if the total derivative exists, it is unique. It also relates the total derivative to directional derivatives.

Theorem 3:

Assume f is differentiable at c with total derivative T_e. Then the directional derivative $f'(c; u)$ exists for every u in R^n and we have $T_e(u) = f'(c; u)$. .... (6)

Proof:

If $v = 0$ then $f'(c; 0) = 0$ and $T_e(0) = 0$. Therefore we can assume that $v \neq 0$. Take $v = hu$ in Taylor's formula (5), with $h \neq 0$, to get

$f(c + hu) - f(c) = T_c(hu) + \| hu \| E_e(v) = hT_c(u) + |h| \, \| u \| E_c(v)$

Now divide by h and let $h \to 0$ to obtain (6).

Theorem 4:

If f is differentiable at c, then f is continuous at c.

Proof:

Let $v \to 0$ in the Taylor formula (5).

The error term $\| v \| E_e(v) \to 0$; the linear term $T_e(v)$ also tends to 0 because

if $v = v_1 u_1 + \cdots + v_n u_n$, where $u_1, \ldots, u_n$ are the unit coordinate vectors,

then by linearity we have $T_e(u) = v_t T_e(u_1) + \cdots + v_n T_e(u_n)$,

and each term on the right tends to 0 as $v \to 0$.

Note:

The total derivative T_e is also written as $f'(c)$ to resemble the notation used in the one-dimensional theory. With this notation, the Taylor formula (5) takes the form

$$f(c+v) = f(c) + f'(c)(v) + \| v \| E_c(v) \qquad \ldots\ldots\ldots\ldots (7)$$

where $E_c(v) \to 0$ as $v \to 0$. However, it should be realized that $f'(c)$ is a linear function, not a number. It is defined everywhere on R^n; the vector $f'(c)(v)$ is the value of $f'(c)$ at q.

Example.

If f is itself a linear function, then $f(c+v) = f(c) + f(v)$, so the derivative f' 'c) exists for every c and equals f. In other words, the total derivative of a linear function is the function itself.

4.5 The Total Derivative Expressed in terms of Partial Derivatives:

The next theorem shows that the vector $f'(c)(v)$ is a linear combination of the partial derivatives of f.

Theorem 5:

Let $f; S \to R^m$ be differentiable at an interior point c of S, where $S \subseteq R^n$. If $v = v_1 n_1 + \cdots + v_n n_n$, where $u_1, \ldots, u_n$ are the unit coordinate vectors in R^n, then

$$f'(c)(v) = \sum_{k=1}^{n} v_k D_k f(c)$$

In particular, if f is real-valued $(m = 1)$ we have $f'(c)(v) = \nabla f(c) \cdot v$, ……….(8)

the dot product of v with the vector $\nabla f(c) = (D_1 f(c), \ldots, D_n f(c))$.

Proof:

We use the linearity of $f'(c)$ to write

$$f'(c)(v) = \sum_{k=1}^{n} f'(c)(v_k u_k) = \sum_{k=1}^{n} v_k f'(c)(u_k)$$

$$= \sum_{k=1}^{n} v_k f'(c; u_k) = \sum_{k=1}^{n} v_k D_k f(c)$$

Note:

The vector $\nabla f(c)$ in (8) is called the gradient vector of f at c. It is defined at each point where the partials $D_1 f, \ldots, D_n f$ exist. The Taylor formula for real-valued f now takes the form

$f(\mathrm{c}+\mathrm{v})=f(\mathrm{c})+\nabla f(\mathrm{c})\cdot\mathrm{v}+o(\|\mathrm{v}\|)$ as $\mathrm{v}\to 0$.

4.6 An Application to Complex-Valued Functions:

Let $f=u+iv$ be a complex-valued function of a complex variable. A necessary condition for f to have a derivative at a point c is that the four partials D_1u, D_2u, D_1v, D_2v exist at c and satisfy the Cauchy-Riemann equations:

$D_1u(c)=D_2b(c),\ D_1v(c)=-D_2u(c)$.

Also, an example showed that the equations by themselves are not sufficient for existence of $f'(c)$. The next theorem shows that the Cauchy-Riemann equations, along with differentiability of u and v, imply existence of $f'(c)$.

Theorem 6:

Let u and v be two real-valued functions defined on a subset S of the complex plane. Assume also that u and v are differentiable at an interior point c of S and that the partial derivatives satisfy the Cauchy-Riemann equations at c. Then the function $f=u+iv$ has a derivative at c. Moreover, $f'(c)=D_1u(c)+iD_1v(c)$.

Proof:

We have $f(z)-f(c)=u(z)-u(c)+i\{v(z)-v(c)\}$ for each z in S. Since each of u and v is differentiable at c, for z sufficiently near to c we have

$u(z)-u(c)=\nabla u(c)\cdot(z-c)+o(\|z-c\|)$

$v(z)-v(c)=\nabla v(c)\cdot(z-c)+o(\|z-c\|)$.

Here we use vector notation and consider complex numbers as vectors in R^2. We then have

$f(z)-f(c)=\{\nabla u(c)+i\nabla v(c)\}\cdot(z-c)+o(\|z-c\|)$

Writing $z=x+iy$ and $c=a+ib$, we find

$$\begin{aligned}&\{\nabla u(c)+i\nabla v(c)\}\cdot(z-c)\\&=D_1u(c)(x-a)+D_2u(c)(y-b)+i\{D_1v(c)(x-a)+D_2v(c)(y-b)\}\\&=D_1u(c)\{(x-a)+i(y-b)\}+iD_1v(c)\{(x-a)+i(y-b)\}\end{aligned}$$

because of the Cauchy-Riemann equations. Hence

$f(z)-f(c)=\{D_1u(c)+iD_1v(c)\}(z-c)+o(\|z-c\|)$.

Dividing by $z-c$ and letting $z\to c$ we see that $f'(c)$ exists and is equal to

$D_1u(c)+iD_1v(c)$.

4.7. The Matrix of a Linear Function:

In this section we digress briefly to record some elementary facts from linear algebra that are useful in certain calculations with derivatives.

Let $T: R^n \to R^n$ be a linear function. (In our applications, T will be the total derivative of a function f.) We will show that T determines an $m \times n$ matrix of scalars (see (9) below) which is obtained as follows:

Let $a_1, \ldots, a_n$, denote the unit coordinate vectors in R^n. If $x \in R^n$ we have $x = x_1 u_1 + \cdots + x_n u_n$ so, by linearity,

$$T(x) = \sum_{k=1}^{n} x_k T(u_k)$$

Therefore T is completely determined by its action on the coordinate vectors $u_1, \ldots, u_n$.

Now let $e_1, \ldots, e_m$ denote the unit coordinate vectors in R^m. Since $T(u_k) \in R^m$, we can write $T(u_k)$ as a linear combination of $e_1, \ldots, e_m$, say

$$T(u_k) = \sum_{i=1}^{m} t_{ik} e_{i-}$$

The scalars $t_l, \ldots, t_{mk}$ are the coordinates of $T(u_k)$. We display these scalars vertically as follows:

$$\begin{bmatrix} t_{1k} \\ t_{2k} \\ \vdots \\ t_{mk} \end{bmatrix}$$

This array is called a column vector. We form the column vector for each of $T(u_1), \ldots, T(u_n)$ and place them side by side to obtain the rectangular array $\begin{bmatrix} t_1 & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix}$ ………..(9)

This is called the matrix of T and is denoted by $m(T)$. It consists of m rows and n columns. The numbers going down the k th column are the components of $T(u_k)$. We also use the notation $m(T) = [t_{ik}]_{i,k=1}^{m,n}$ or $m(T) = (t_{ik})$ to denote the matrix in (9).

Now let $T: R^n \rightarrow R^m$ and $S: R^m \rightarrow R^p$ be two linear functions, with the domain of S containing the range of T. Then we can form the composition $S \cdot T$ defined by

$(S \circ T)(x) = S[T(x)]$ for all x in R^x.

The composition $S \circ T$ is also linear and it maps R^n into R^p.

Let us calculate the matrix $m(S \circ T)$. Denote the unit coordinate vectors in R^n, R^m, and R^p, respectively, by

$u_1, \ldots, u_n, \; e_1, \ldots, e_m,$ and $w_1, \ldots, w_p$.

Suppose that S and T have matrices (s_{ij}) and (t_{ij}), respectively. This means that

and $S(e_k) = \sum_{i=1}^{2} s_{ik} w_i$ for $k = 1,2, \ldots, m$

Then $T(u_j) = \sum_{k=1}^{m} t_{kj} e_k$ for $j = 1,2, \ldots, n$.

$$(S \circ T)(u_j) = S[T(u_j)] = \sum_{k=1}^{m} t_k S(e_k) = \sum_{k=1}^{m} t_{kj} \sum_{i=1}^{p} s_{ik} w_i$$

$$= \sum_{i=1}^{p} \left(\sum_{k=1}^{m} s_{ik} t_{kj} \right) w_i$$

So $m(S \circ T) = \left[\sum_{k=1}^{m} s_{ik} t_{kj}\right]_{i,j-1}^{p,n}$.

In other words, $m(S \circ T)$ is a $p \times n$ matrix whose entry in the i th row and j th column is $\sum_{k=1}^{m} s_{ik} t_{kj}$

the dot product of the i th row of $m(S)$ with the j th column of $m(T)$. This matrix is also called the product $m(S)m(T)$. Thus, $m(S \circ T) = m(S)m(T)$.

4.8 The Jacobian Matrix:

Next we show how matrices arise in connection with total derivatives.

Let f be a function with values in R^m which is differentiable at a point c in R^n, and let $T = f'(c)$ be the total derivative of f at c. To find the matrix of T we consider its action on the unit coordinate vectors $u_1, \ldots, u_n$. By Theorem 3 we have

$T(u_k) = f'(c; u_k) = D_k f(c)$.

To express this as a linear combination of the unit coordinate vectors $e_1, \ldots, e_m$ of R^m we write $f = (f_1, \ldots, f_m)$ so that $D_k f = (D_k f_1, \ldots, D_k f_m)$, and hence

$$T(u_k) = D_k f(c) = \sum_{i=1}^{m} D_k f_i(c) c_i$$

Therefore the matrix of T is $m(T) = (D_k f_i(c))$. This is called the Jacobian matrix of f at c and is denoted by Df(c). That is, $Df(c) = \begin{bmatrix} D_1 f_1(c) & D_2 f_1(c) & \cdots & D_n f_1(c) \\ D_1 f_2(c) & D_2 f_2(c) & \cdots & D_n f_2(c) \\ \vdots & \vdots & & \vdots \\ D_1 f_m(c) & D_2 f_m(c) & \cdots & D_n f_m(c) \end{bmatrix}$ ……….(10)

The entry in the i th row and k th column is $D_k f_i(c)$. Thus, to get the catries in the k th column, differentiate the components of f with respect to the k th coordinate vector. The Jacobian matrix Df(c) is defined at each point c in R^n where all the partial derivatives $D_k f_i(c)$ exist.

The k th row of the Jacobian matrix (10) is a vector in R^n called the gradient rector of f_k, denoted by $\nabla f_k(c)$. That is, $\nabla f_k(c) = (D_1 f_k(c), \dots, D_n f_k(c))$

In the special case when f is real-valued $(m = 1)$, the Jacobian matrix consists of only one row. In this case $Df(c) = \nabla f(c)$, and Equation (8) of Theorm 12.5 shows that the directional derivative $f'(c; v)$ is the dot product of the gradient vector $\nabla f(c)$ with the direction v.

For a vector-valued function $f = (f_1, \dots, f_m)$

we have $f'(c)(v) = f'(c; v) = \sum_{k=1}^{m} f_k'(c; v) e_k = \sum_{k=1}^{m} \{\nabla f_k(c) \cdot v_j\} e_k$. ………(11)

so the vector $f'(c)(v)$ has components $(\nabla f_1(c) \cdot v, \dots, \nabla f_m(c) \cdot v)$

Thus, the components of $f'(c)(v)$ are obtained by taking the dot product of the successive rows of the Jacobian matrix with the vector v. If we regard $f'(c)(v)$ as an $m \times 1$ matrix, or column vector, then $f'(c)(v)$ is equal to the matrix product Df(c)v, where Df(c) is the $m \times n$ Jacobian matrix and v is regarded as an $n \times 1$ matrix, or column vector.

Note:

Equation (11), used in conjunction with the triangle inequality and the Cauchy-Schwarz inequality, gives us $\|f'(c)(v)\| = \|\sum_{k=1}^{m} \{\nabla f_k(c) \cdot v\} e_k\| \leq \sum_{k=1}^{m} |\nabla f_k(c) \cdot v| \leq \| v \| \sum_{k=1}^{m} \|\nabla f_k(c)\|$.

Therefore we have $\|f'(c)(v)\| \leq M \| v \|$, ………..(12)

where $M = \sum_{k=1}^{m} \|\nabla f_k(c)\|$. This inequality will be used in the proof of the chain rule. It also shows that $f'(c)(v) \to 0$ as $v \to 0$.

4.9 The Chain Rule:

Let f and g be functions such that the composition $h = f \circ g$ is defined in a neighborhood of a point α. The chain rule tells us how to compute the total derivative of h in terms of total derivatives of f and of g.

Theorem 12.7:

Assume that g is differentiable at a, with total derivative $g'(a)$. Let $b = g(a)$ and assume that I is differentiable at b, with total derivative $I'(b)$. Then the composite function $h = f \circ g$ is differentiable at a, and the total derivative $h'(a)$ is given by $h'(a) = f'(b) \circ g'(a)$, the composition of the linear functions $f'(b)$ and $g'(a)$.

Proof:

We consider the difference $h(a + y) - h(a)$ for small $\| y \|$, and show that we have a first-order Tayior formula. We have $h(a + y) - h(a) = f[g(a + y)] - f[g(a)] = f(b + v) - f(b)$, ……(13)

where $b = g(a)$ and $v = g(a + y) - b$. The Taylor formula for $g(a + y)$ implies

$$v = g'(a)(y) + \| y \| E_s(y), \text{ where } E_q(y) \to 0 \text{ as } y \to 0.$$

The Taylor formula for $f(b + v)$ implies

$$f(b + v) - f(b) = f'(b)(v) + \| v \| E_b(v), \text{ where } E_b(v) \to 0 \text{ as } v \to 0. \qquad (15)$$

Using equation (14) in (15) we find

$f(b + v) - f(b) = f'(b)[g'(a)(y)] + f'(b)[\| y \| E_a(y)] + \| v \| E_b(v)$

where $E(0) = 0$ and $E(y) = f'(b)[E_a(y)] + \frac{\|v\|}{\|y\|} E_b(v)$ if $y \neq 0$. ……….(17)

To complete the proof we need to show that $E(y) \to 0$ as $y \to 0$.

The first term on the right of (17) tends to 0 as $y \to 0$ because $E_2(y) \to 0$. In the second term, the factor $E_z(v) \to 0$ because $v \to 0$ as $y \to 0$. Now we show that the quotient $\| v \| / \| y \|$ remains bounded as $y \to 0$. Using (14) and (12) to estimate the numerator we find

$\| v \| \leq \|g'(a)(y)\| + \| y \| \|E_a(y)\| \leq \| y \| \{M + \|E_s(y)\|\}$,

where $M = \sum_{k=1}^{m} \|\nabla g_k(a)\|$. Hence

$$\frac{\| v \|}{\| y \|} \leq M + \|E_a(y)\|$$

so $\| v \| / \| y \|$ remains bounded as $y \to 0$. Using (13) and (16) we obtain the Taylor formula

$h(a + y) - h(a) = f'(b)[g'(a)(y)] + \| y \| E(y)$, where $E(y) \to 0$ as $y \to 0$. This proves that h is differentiable at and that its total derivative at a is the composition $f'(b) \circ g'(a)$.

4.10 Matrix form of the Chain Rule:

The chain rule states that $h'(a) = f'(b) \circ g'(a)$ …………(18)

where $h = f \circ g$ and $b = g(a)$. Since the matrix of a composition is the product of the corresponding matrices, (18) implies the following relation for Jacobian matrices:

$Dh(a) = Df(b)Dg(z).$ …………. (19)

This is called the matrix form of the chain rule. It can also be written as a set of scalar equations by expressing each matrix in terms of its entries.

Specifically, suppose that $a \in R^p, b = g(a) \in R^n$, and $f(b) \in R^m$. Then $h(a) \in R^m$ and we can write $g = (g_1, \dots, g_n)$, $f = (f_1, \dots, f_m)$, $h = (h_1, \dots, h_m)$.

Then $Dh(a)$ is an $m \times p$ matrix, $Df(b)$ is an $m \times n$ matrix, and $Dg(a)$ is an $n \times p$ matrix, given by $Dh(a) = \left[D_j h_i(a)\right]_{i,j=1}^{m,p}$, $Df(b) = [D_k f_i(b)]_{i,k=1}^{m,n}$, $Dg(a) = \left[D_j g_k(a)\right]_{k,j=1}^{n,p}$.

The matrix equation (19) is equivalent to the mp scalar equations

$D_j h_i(a) = \sum_{k=1}^{n} D_k f_k(b) D_j g_k(a)$, for $i = 1,2,\dots,m$ and $j = 1,2,\dots,p$.

These equations express the partial derivatives of the components of h in terms of the partial derivatives of the components of f and g.

The equation in (20) can be put in a form that is easier to remember. Write $y = f(x)$ and $x = g(t)$.

Then $y = f[g(t)] = h(t)$, and (20) becomes $\frac{\partial y_i}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial x_k}{\partial t_j}$ ………(21)

Where $\frac{\partial y_i}{\partial t_j} = D_j h_j$, $\frac{\partial y_i}{\partial x_k} = D_k f_i$, and $\frac{\partial x_k}{\partial t_j} = D_j g_k$

Examples. Suppose $m = 1$. Then both f and $h = f \circ g$ are real-valued and there are p equations in (20), one for each of the partial derivatives of h :

$$D_j h(a) = \sum_{k=1}^{n} D_k f(b) D_s g_k(a),\ j = 1,2,\dots,p$$

The right member is the dot product of the two vectors $\nabla f(b)$ and $D_j g(a)$. In this case Equation (21) takes the form

$$\frac{\partial y}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial y}{\partial x_k}\frac{\partial x_k}{\partial t_j},\ j = 1,2, \ldots, p.$$

In particular, if $p = 1$ we get only one equation,

$$h'(\mathrm{a}) = \sum_{k=1}^{n} D_k f(\mathrm{b}) g_k'(\mathrm{a}) = \nabla f(\mathrm{b}) \cdot \mathrm{Dg(a)}$$

where the Jacobian matrix Dg(a) is a column vector.

The chain rule can be used to give a simple proof of the following theorem for differentiating an integral with respect to a parameter which appears both in the integrand and in the limits of integration.

Theorem 12.8:

Let f and $D_2 f$ be contimuous on a rectangle $[a, b] \times [c, d]$. Let p and q be differentiable on $[c, d]$, where $p(y) \in [a, b]$ and $q(y) \in [a, b]$ for each y in $[c, d]$. Define F by the equation

$$F(y) = \int_{p(y)}^{Q(y)} f(x, y)dx, \text{ if } y \in [c, d].$$

Then $F'(y)$ exists for each y in (c, d) and is given by

$$F'(y) = \int_{p(y)}^{q(y)} D_2 f(x; y)dx + f(q(y), y)q'(y) - f(p(y), y)p'(y).$$

Proof:

Let $G(x_1, x_2, x_3) = \int_{x_1}^{x_2} f(t, x_3)dt$ whenever x_1 and x_2 are in $[a, b]$ and $x_3 \in [c, d]$. Then F is the composite function given by $F(y) = G(p(y), q(y), y)$. The chain rule implies

$F'(y) = D_1 G(p(y), q(y), y)p'(y) + D_2 G(p(y), q(y), y)q'(y) + D_3 G(p(y), q(y), y)$.

By Theorem 7.32, we have $D_1 G(x_1, x_2, x_3) = -f(x_1, x_3)$ and $D_2 G(x_1, x_2, x_3) = f(x_2, x_3)$. By

$$D_3 G(x_1, x_2, x_3) = \int_{x_3}^{x_2} D_2 f(t, x_3)dt$$

Using these results in the formula for $F'(y)$ we obtain the theorem.

4.11 The Mean-Value Theorem for Differentiable Functions

The Mean-Value Theorem for functions from R^1 to R^1 states that

$f(y) - f(x) = f'(z)(y - x)$, ………….. (22)

where z lies between x and y. This equation is false, in general, for vector-valued functions from R^x to R^m, when $m > 1$. (See Exercise 12.19.) However, we will show that a correct equation is obtained by taking the dot product of each member of (22) with any vector in R^m, provided z is suitably chosen. This gives a useful generalization of the Mean-Value Theorem for vector-valued functions.

In the statement of the theorem we use the notation $L(x, y)$ to denote the line segment joining two points x and y in R^n. That is,

$L(x, y) = \{tx + (1 - t)y: 0 \leq t \leq 1\}$.

Theorem 9 (Mean-Value Theorem.):

Let S be an open subset of R^n and assume that $f: S \to R^m$ is differentiable at each point of S. Let x and y be two points in S such that $L(x, y) \subseteq S$. Then for every vector a in R^m there is a point z in $L(x, y)$ such that $a \cdot \{f(y) - f(x)\} = a \cdot \{f'(z)(y - x)\}$. …………..(23)

Proof:

Let $u = y - x$. Since S is open and $L(x, y) \subseteq S$, there is a $\delta > 0$ such that $x + tu \in S$ for all real t in the interval $(-\delta, 1 + \delta)$. Let a be a fixed vector in R^m and let F be the real-valued function defined on $(-\delta, 1 + \delta)$ by the equation

$F(t) = a \cdot f(x + tu)$.

Then F is differentiable on $(-\delta, 1 + \delta)$ and its derivative is given by

$F'(t) = a \cdot f'(x + tu; u) = a \cdot \{f'(x + tu)(u)\}$

By the usual Mean-Value Theorem we have

Now

$F(1) - F(0) = F'(\theta)$, where $0 < \theta < 1$.

$F'(\theta) = a \cdot \{f'(x + \theta u)(u)\} = a \cdot \{f'(z)(y - x)\}$,

where $z = x + \theta\, u \in L(x, y)$. But $F(1) - F(0) = a \cdot \{f(y) - f(x)\}$, so we obtain (23). Of course, the point z depends on F, and hence on a.

Note:

If S is convex, then $L(\mathrm{x}, \mathrm{y}) \subseteq S$ for all x, y in S so (23) holds for all x and y in S.

Examples:

1. If f is real-valued $(m = 1)$ we can take $a = I$ in (23) to obtain
$f(\mathrm{y}) - f(\mathrm{x}) = f'(\mathrm{z})(\mathrm{y} - \mathrm{x}) = \nabla f(\mathrm{z}) \cdot (\mathrm{y} - \mathrm{x})$. ………… (24)
2. If f is vector-valued and if a is a unit vector in $R^m, \| a^{\ m} = 1, Eq.$ (23) and the Cauchy Schwarz inequality give us $\| f(y) - f(x) \| \leq \|f'(z)(y - x)\|$. Using (12) we obtain the inequality $\| f(y) - f(x) \| \leq M \| y - x \|$, where $M = \sum_{k=1}^{m} | \nabla f_k(z) \|$. Note that M depends on z and hence on x and y.
3. If S is convex and if all the partial derivatives $D_f f_k$ are bounded on S, then there is a constant $A > 0$ such that $\| \mathrm{f(y)} - \mathrm{f(x)} \| \leq A \| \mathrm{y} - \mathrm{x} \|$. In other words, f satisfies a Lipschitz condition on S.

The Mean-Value Theorem gives a simple proof of the following result concerning functions with zero total derivative.

Theorem 10:

Let S be an open connected subset of R^n, and let $\mathrm{f}: S \to \mathrm{R}^m$ be differentiable at each point of S. If $\mathrm{f}'(\mathrm{c}) = 0$ for each c in S, then f is constant on S.

Proof:

Since S is open and connected, it is polygonally connected. Therefore, every pair of points x and y in S can be joined by a polygonal arc lying in S. Denote the vertices of this arc by $\mathrm{p}_1, \dots, \mathrm{p}_r$, where $\mathrm{p}_1 = \mathrm{x}$ and $\mathrm{Pr} = \mathrm{y}$. Since each segment $L(\mathrm{p}_{i+1}, \mathrm{p}_i) \subseteq S$, the Mean-Value Theorem shows that $\mathrm{a} \cdot \{\mathrm{f}(\mathrm{p}_{i+1}) - \mathrm{f}(\mathrm{p}_i)\} = 0$,

for every vector a. Adding these equations for $i = 1,2, \dots, r - 1$, we find $\mathrm{a} \cdot \{\mathrm{f(y)} - \mathrm{f(x)}\} = 0$, for every a. Taking $\mathrm{a} = \mathrm{f(y)} - \mathrm{f(x)}$ we find $\mathrm{f(x)} = \mathrm{f(y)}$, so f is constant on S.

4.12. A Sufficient Condition for Differentiability:

Theorem 11:

Assume that one of the partial derivatives $D_1\mathrm{f}, \ldots, D_n\mathrm{f}$ exists at c and that the remaining $n-1$ partial derivatives exist in some n-ball $B(\mathrm{c})$ and are continuous at c. Then f is differentiable at c.

Proof:

First we note that a vector-valued function $\mathrm{f} = (f_1, \ldots, f_m)$ is differentiable at c if, and only if, each component f_k is differentiable at c. (The proof of this is an easy exercise.) Therefore, it suffices to prove the theorem when f is real-valued.

For the proof we suppose that $D_1 f(c)$ exists and that the continuous partials are $D_2 f, \ldots, D_n f$.

The only candidate for $f'(\mathrm{c})$ is the gradient vector $\nabla f(\mathrm{c})$. We will prove that

$f(\mathrm{c}+\mathrm{v}) - f(\mathrm{c}) = \nabla f(\mathrm{c}) \cdot \mathrm{v} + o(\| \mathrm{v}i)$ as $\mathrm{v} \to 0$,

and this will prove the theorem. The idea is to express the difference $f(\mathrm{c}+\mathrm{v}) - f(\mathrm{c})$ as a sum of n terms, where the k th term is an approximation to $D_k f(\mathrm{c}) v_k$.

For this purpose we write $\mathrm{v} = \lambda y$, where $\| \mathrm{y} \| = 1$ and $\lambda = \| \mathrm{v} \|$. We keep λ small enough so that $\mathrm{c} + \mathrm{y}$ lies in the ball $B(\mathrm{c})$ in which the partial derivatives $D_2 f, \ldots, D_n f$ exist. Expressing y in terms of its components we have $\mathrm{y} = y_1 \mathrm{b}_1 + \cdots + y_n \mathrm{u}_n$,

where $\mathrm{u_k}$ is the k th unit coordinate vector. Now we write the difference $f(\mathrm{c}+\mathrm{v}) - f(\mathrm{c})$ as a telescoping sum, $f(\mathrm{c}+\mathrm{v}) - f(\mathrm{c}) = f(\mathrm{c}+\lambda \mathrm{y}) - f(\mathrm{c}) = \sum_{k=1}^{n} \{f(\mathrm{c}+i\mathrm{v_k}) - f(\mathrm{c}+i\mathrm{v}_{k-1})\}$,

Where $\mathrm{v}_0 = 0$, $\mathrm{v}_1 = y_1 \mathrm{u}_1$, $\mathrm{v}_2 = y_1 \mathrm{u}_1 + y_2 \mathrm{u}_2, \ldots, \mathrm{v}_n = y_1 \mathrm{u}_1 + \cdots + y_n \mathrm{u}_n$.

The first term in the sum is $f(\mathrm{c} + \lambda y_1 \mathrm{w}_1) - f(\mathrm{c})$. Since the two points c and $\mathrm{c} + \lambda y_1 \mathrm{u}_1$ differ only in their first component, and since $D_1 f(\mathrm{c})$ exists, we can write

$f(\mathrm{c} + \lambda y_1 \mathrm{a}_1) - f(\mathrm{c}) = \lambda y_1 D_1 f(\mathrm{c}) + \lambda y_1 E_1(\lambda)$,

where $E_1(\lambda) \to 0$ as $\lambda \to 0$.

For $k \geq 2$, the k th term in the sum is

$f(\mathrm{c} + \lambda \mathrm{v}_{k-1} + \lambda y_k \mathrm{u}_k) - f(\mathrm{c} + \lambda \mathrm{v}_{k-1}) = f(\mathrm{b}_k + \lambda y_k \mathrm{u}_k) - f(\mathrm{b}_k)$,

where $\mathrm{b}_k = \mathrm{c} + \lambda v_{k-1}$. The two points b_k and $\mathrm{b}_k + \lambda y_k \mathrm{u_k}$ differ only in their k th component, and we can apply the one-dimensional Mean-Value Theorem for derivatives to write $f(\mathrm{b}_k + \lambda y_k \mathrm{w}_k) - f(\mathrm{b}_k) = \lambda y_k D_k f(\mathrm{a}_k)$, ………..(26)

where a_k lies on the line segment joining b_k to $b_k + \lambda y_k n_k$. Note that $b_k \to c$ and hence $n_k \to c$ as $\lambda \to 0$. Since each $D_k f$ is continuous at c for $k \geq 2$ we can write

$D_k f(a_k) = D_k f(c) + E_k(\lambda),.$ where $E_k(\lambda) \to 0$ as $\lambda \to 0$.

Using this in (26) we find that (25) becomes

$$f(c + v) - f(c) = \lambda \sum_{k=1}^{n} D_k f(c) y_k + \lambda \sum_{k=1}^{n} y_k E_k(\lambda)$$

$$= \nabla f(c) \cdot v + \| v \| E(\lambda),$$

Where $E(\lambda) = \sum_{k=1}^{n} y_k E_k(\lambda) \to 0$ as $\| v \| \to 0$

Note:

Continuity of at least $n - 1$ of the partials $D_1 f, \ldots, D_n f$ at c, although sufficient, is by no means necessary for differentiability of f at c.

4.13. A Sufficient Condition for Equality of Mixed Partial Derivatives:

The partial derivatives $D_1 f, \ldots, D_n f$ of a function from R^n to R^m are themselves functions from R^n to R^m and they, in turn, can have partial derivatives. These are called second-order partial derivatives. We use the notation introduced in Chapter 5 for real-valued functions:

$$D_{r,k} f = D_r(D_k f) = \frac{\partial^2 f}{\partial x_r \, \partial x_k}$$

Higher-order partial derivatives are similarly defined.

The example $f(x, y) = \begin{cases} xy(x^2 - y^2)/(x^2 + y^2) & \text{if } (x, y) \neq (0,0), \\ 0 & \text{if } (x, y) = (0,0), \end{cases}$

shows that $D_{1,2} f(x, y)$ is not necessarily the same as $D_{2,1} f(x, y)$. In fact, in this example we have

$$D_1 f(x, y) = \frac{y(x^4 + 4x^2 y^2 - y^4)}{(x^2 + y^2)^2}, \quad \text{if } (x, y) \neq (0,0)$$

and $D_1 f(0,0) = 0$. Hence, $D_1 f(0, y) = -y$ for all y and therefore

$D_{2,1} f(0, y) = -1,\ D_{2,1} f(0,0) = -1.$

On the other hand, we have $D_2 f(x, y) = \frac{x(x^4 - 4x^2 y^2 - y^4)}{(x^2 + y^2)^2}$, if $(x, y) \neq (0,0)$,

and $D_2 f(0,0) = 0$, so that $D_2 f(x, 0) = x$ for all x. Therefore, $D_{1,2} f(x, 0) = 1$, $D_{1,2} f(0,0) = 1$, and we see that $D_{2,1} f(0,0) \neq D_{1,2} f(0,0)$.

The next theorem gives us a criterion for determining when the two mixed partials $D_{1,2}$f and $D_{2,1}$f will be equal.

Theorem 12:

If both partial derivatives D_1f and D_2f exist in an n-ball $B(\text{c}; \delta)$ and if both are differentiable at c, then $D_{r,\text{f}} f(\text{c}) = D_{\text{k,r}} f(\text{c})$. ....(27)

Proof.

If f $= (f_1, \dots, f_m)$, then D_kf $= (D_k f_1, \dots, D_k f_m)$. Therefore it suffices to prove the theorem for real-valued f. Also, since only two components are involved in (27), it suffices to consider the case $n = 2$. For simplicity, we assume that c $= (0,0)$. We shall prove that

$D_{1,2} f(0,0) = D_{2,1} f(0,0)$.

Choose $h \neq 0$ so that the square with vertices $(0,0), (h, 0), (h, h)$, and $(0, h)$ lies in the 2 -ball $B(0; \delta)$. Consider the quantity

$\Delta(h) = f(h, h) - f(h, 0) - f(0, h) + f(0,0)$.

We will show that $\Delta(h)/h^2$ tends to both $D_{2,1} f(0,0)$ and $D_{1,2} f(0,0)$ as $h \to 0$.

Let $G(x) = f(x, h) - f(x, 0)$ and note that $\Delta(h) = G(h) - G(0)$. ....(28)

By the one-dimensional Mean-Value Theorem we have

$G(h) - G(0) = hG'(x_1) = h\{D_1 f(x_1, h) - D_t f(x_1, 0)\}$, ... (29)

where x_1 lies between 0 and h. Since D, f is differentiable at (0,0), we have the first-order Taylor formulas

$D_1 f(x_1, h) = D_1 f(0,0) + D_{1,2} f(0,0) x_1 + D_{2,1} f(0,0) h + (x_1^2 + h^2)^{1/2} E_1(h)$,

And $D_1 f(x_1, 0) = D_1 f(0,0) + D_{1,1} f(0,0) x_1 + |x_1| E_2(h)$,

where $E_1(h)$ and $E_2(h) \to 0$ as $h \to 0$. Using these in (29) and (28) we find

$\Delta(h) = D_{2,1} f(0,0) h^2 + E(h)$

where $E(h) = h(x_1^2 + h^2)^{1/2} E_1(h) + h|x_1| E_2(h)$. Since $|x_1| \leq |h|$, we have

$0 \leq |E(h)| \leq \sqrt{2} h^2 |E_1(h)| + h^2 |E_2(h)|$,

So $\lim_{h \to 0} \frac{\Delta(h)}{h^2} = D_{2,1} f(0,0)$

Applying the same procedure to the function $H(y) = f(h, y) - f(0, y)$ in place of $G(x)$, we find that $\lim_{h \to 0} \frac{\Delta(h)}{h^2} = D_{1.2} f(0,0)$

As a consequence of Theorems 11 and 12 we have:

Theorem 13:

If both partial derivatives $D_{\mathrm{r}}\mathrm{f}$ and $D_2\mathrm{f}$ exist in an n-ball $B(\mathrm{c})$ and if both $D_{r,k}\mathrm{f}$ and $D_{t,r}\mathrm{f}$ are continuous at c, then $D_{r,d}f(\mathrm{c}) = D_{k,r}f(\mathrm{c})$

Note:

We mention (without proof) another result which states that if $D_k, D_k f$ and $D_{k,r}\mathrm{f}$ are continuous in an n-ball $B(\mathrm{c})$, then $D_{r,k}f(c)$ exists and equals $D_{k,f}f(c)$.

If f is a real-valued function of two variables, there are four second-order partial derivatives to consider; namely, $D_{1,1}f, D_{1,2}f, D_{2,1}f$, and $D_{2,2}f$. We have just shown that only three of these are distinct if f is suitably restricted.

The number of partial derivatives of order k which can be formed is 2^k. If all these derivatives are continuous in a neighborhood of the point (x, y), then certain of the mixed partials will be equal. Each mixed partial is of the form $D_{r_1}, \ldots, r_k f$, where each r_j is either 1 or 2 . If we have two such mixed partials, $D_{r_1}, \ldots, r_k f$ and $D_{p_1}, \ldots, p_{p_k} f$, where the k-tuple $(r_1, \ldots, r_k)$ is a permutation of the k-tuple $(p_1, \ldots, p_k)$, then the two partials will be equal at (x, y) if all 2^k partials are continuous in a neighborhood of (x, y). This statement can be easily proved by mathematical induction, using Theorem 13 (which is the case $k = 2$). We omit the proof for general k. From this it follows that among the 2^k partial derivatives of order k, there are only $k + 1$ distinct partials in general, namely, those of the form $D_{r_1}, \ldots, r_k f$, where the k-tuple $(r_1, \ldots, r_k)$ assumes the following $k + 1$ forms:

$(2,2, \ldots ,2),\ (1,2,2, \ldots ,2),\ (1,1,2, \ldots ,2), \ldots,\ (1,1, \ldots ,1,2),\ (1, \ldots ,1)$.

Similar statements hold, of course, for functions of n variables. In this case, there are n^k partial derivatives of, order k that can be formed. Continuity of all these partials at a point x implies that $D_{r_1}, \ldots, r_x f(\mathrm{x})$ is unchanged when the indices $r_1, \ldots, r_k$ are permuted. Each r_i is now a positive integer $\leq n$.

4.14. Taylor's Formula for Functions from R^* To R^1:

Taylor's formula can be extended to real-valued functions f defined on subsets of R^n. In order to state the general theorem in a form which resembles the one-dimensional case, we introduce special symbols $f''(x;t), f'''(x;t), \ldots, f^{(m)}(x;t)$,for certain sums that arise in Taylor's formula. These play the role of higher order directional derivatives, and they are defined as follows:

If x is a point in R^* where all second-order partial derivatives of f exist, and if $t = (t_1, \ldots, t_n)$ is an arbitrary point in R^*, we write

We also define $f''(x;t) = \sum_{j=1}^{n} \sum_{j=1}^{n} D_{i,j} f(x) t_j t_t$

$$f''(x;t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} D_{i,j,k} f(x) t_k t_j t_i$$

if all third-order partial derivatives exist at x. The symbol $f^{(m)}(x;t)$ is similarly defined if all mth-order partials exist.

These sums are analogous to the formula $f'(x;t) = \sum_{i=1}^{n} D_D f(x) t_i$

for the directional derivative of a function which is differentiable at x.

Theorem 14 (Taylor's formula):

Assume that f and all its partial derivatives of order $< m$ are differentiable at each point of an open set S in R^n. If a and b are two points of S such that $L(a,b) \subseteq S$, then there is a point z on the line segment $L(a,b)$ such that $f(b) - f(a) = \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(a; b-a) + \frac{1}{m!} f^{(m)}(z; b-a)$.

Proof:

Since S is open, there is a $\delta > 0$ such that $a + t(b-a) \in S$ for all real t in the interval $-\delta < t < 1 + \delta$. Define g on $(-\delta, 1+\delta)$ by the equation $g(t) = f[a + t(b-a)]$.

Then $f(b) - f(a) = g(1) - g(0)$. We will prove the theorem by applying the one-dimensional Taylor formula to g, writing

$g(1) - g(0) = \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(0) + \frac{1}{m!} g^{(m)}(\theta)$, where $0 < \theta < 1$. ……….. (30)

Now g is a composite function given by $g(t) = f[p(t)]$, where $p(t) = a + t(b-a)$. The k th component of p has derivative $p_k'(t) = b_k - a_k$. Applying the chain rule, we see that $g'(t)$ exists in the interval $(-\delta, 1+\delta)$ and is given by the formula

$$g'(t) = \sum_{j=1}^{n} D_j f[\mathrm{P}(t)](b_j - a_j) = f'(\mathrm{p}(t); \mathrm{b} - \mathrm{a}).$$

Again applying the chain rule, we obtain

$$g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f[\mathrm{p}(t)](b_j - a_j)(b_i - a_i) = f''(\mathrm{p}(t); \mathrm{b} - \mathrm{a}).$$

Similarly, we find that $g^{(m)}(t) = f^{(m)}(p(t); \mathrm{b} - \mathrm{a})$. When these are used in (30) we obtain the theorem, since the point $\mathrm{z} = \mathrm{a} + \theta(\mathrm{b} - \mathrm{a}) \in L(\mathrm{a}, \mathrm{b})$.

Unit V

Implicit Functions and Extremum Problems: Functions with non-zero Jacobian determinants – The inverse function theorem-The Implicit function theorem -Extrema of real valued functions of severable variables -Extremum problems with side conditions.

Chapter 5: Sections 5.1- 5.7

Implicit Functions and Extremum Problems

5.1 Introduction:

This chapter consists of two principal parts. The first part discusses an important theorem of analysis called the implicit function theorem; the second part treats extremum problems. The implicit function theorem in its simplest form deals with an equation of the form

$f(x,t) = 0.$ ……….. (1)

The problem is to decide whether this equation determines x as a function of t. If so, we have $x = g(t)$, for some function g. We say that g is defined "implicitly" by (1).

The problem assumes a more general form when we have a system of several equations involving several variables and we ask whether we can solve these equations for some of the variables in terms of the remaining variables. This is the same type of problem as above, except that x and t are replaced by vectors, and f and g are replaced by vector-valued functions. Under rather general conditions, a solution always exists. The implicit function theorem gives a description of these conditions and some conclusions about the solution.

An important special case is the familiar problem in algebra of solving n linear equations of the form $\sum_{j=1}^{n} a_{ij}x_j = t_i \ (i = 1,2,\dots n)$ ………(2)

where the a_{ij} and t_i are considered as given numbers and $x_1, \dots, x_n$ represent unknowns. In tinear algebra it is shown that such a system has a unique solution if, and only if, the determinant of the coefficient matrix $A = [a_{ij}]$ is nonzero.

Note:

The determinant of a square matrix $A = [a_{ij}]$ is denoted by $\det A$ or $\det [a_{ij}]$. If $\det [a_{ij}] \neq 0$, the solution of (2) can be obtained by Cramer's rule which expresses each x_k as a quotient of two determinants, say $x_k = A_k/D$, where $D = \det[a_{ij}]$ and A_k is the determinant of the matrix obtained by replacing the k th column of $[a_{ij}]$ by $i_1, \ldots, t_n$. In particular, if each $t_i = 0$, then each $x_k = 0$. Next we show that the system (2) can be written in the form (1). Each equation in (2) has the form $f_i(\mathrm{x}, \mathrm{t}) = 0$ where $\mathrm{x} = (x_1, \ldots, x_n)$, $\mathrm{t} = (t_1, \ldots, t_n)$, and $f(\mathrm{x}, \mathrm{t}) = \sum_{j=1}^{n} a_{ij}x_j - t_i$. Therefore the system in (2) can be expressed as one vector equation $f(x, t) = 0$, where $\mathrm{f} = (f_1, \ldots, f_n)$. If $D_j f_i$ denotes the partial derivative of f_i with respect to the j th coordinate x_j, then $D_j f_i(\mathrm{x}, \mathrm{t}) = a_{ij}$. Thus the coefficient matrix $A = [a_{ij}]$ in (2) is a Jacobian matrix. Linear algebra tells us that (2) has a unique solution if the determinant of this Jacobian matrix is nonzero.

In the general implicit function theorem, the non -vanishing of the determinant of a Jacobian matrix also plays a role. This comes about by approximating f by a linear function. The equation $\mathrm{f}(\mathrm{x}, \mathrm{t}) = 0$ gets replaced by a system of linear equations whose coefficient matrix is the Jacobian matrix of f.

Notation:

If $\mathrm{f} = (f_1, \ldots, f_n)$ and $\mathrm{x} = (x_1, \ldots, x_n)$, the Jacobian matrix $\mathrm{D}(\mathrm{x}) = [D_j f(x)]$ is an $n \times n$ matrix. Its determinant is called a Jacobian determinant and is denoted by $J_{\mathrm{f}}(\mathrm{x})$. Thus,

$J_{\mathrm{r}}(\mathrm{x}) = \det \mathrm{Df}(\mathrm{x}) = \det[D_j f_i(\mathrm{x})]$. The notation $\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$ is also used to denote the Jacobian determinant $J_t(x)$. The next theorem relates the Jacobian determinant of a complex-valued function with its derivative.

Theorem 1:

If $f = u + iv$ is a complex-valued function with a derivative at a point z in C, then $J_f(z) = |f'(z)|^2$.

Proof:

We have $f'(z) = D_1 u + i D_1 v$, so $|f'(z)|^2 = (D_1 u)^2 + (D_1 v)^2$. Also,

$$J_S(z) = \det \begin{bmatrix} D_1 u & D_2 u \\ D_1 v & D_2 v \end{bmatrix} = D_1 u D_2 v - D_1 v D_2 u = (D_1 u)^2 + (D_2 v)^2,$$

by the Cauchy-Riemann equations.

5.2 Functions with Nonzero Jacobian Determinant:

This section gives some properties of functions with nonzero Jacobian determinant at certain points. These results will be used later in the proof of the implicit function theorem.

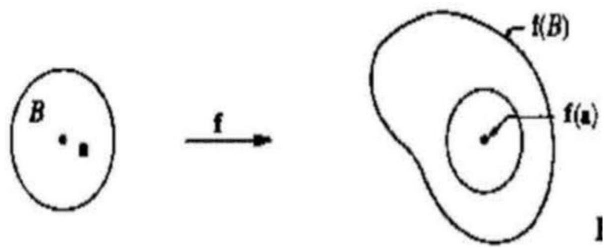

Figure 5.1

Theorem 2:

Let $B = B(\mathrm{a}; r)$ be an n-ball in R^n, let ∂B denote its boundary, $\partial B = \{\mathrm{x}: \| \mathrm{x} - \mathrm{a} \| = r\}$, and let $\bar{B} = B \cup \partial B$ denote its closure. Let $\mathrm{f} = (f_1, \dots, f_n)$ be continuous on $\bar{B}$, and assume that all the partial derivatives $D_j f_i(\mathrm{x})$ exist if $\mathrm{x} \in B$. Assume further that $\mathrm{f(x)} \neq \mathrm{f(a)}$ if $\mathrm{x} \in \partial B$ and that the Jacobian determinant $J_{\mathrm{f}}(\mathrm{x}) \neq 0$ for each x in B. Then $\mathrm{f}(B)$, the image of B under f, contains an n-ball with center at $\mathrm{f(a)}$.

Proof:

Define a real-valued function g on ∂B as follows: $g(\mathrm{x}) = \| \mathrm{f(x)} - \mathrm{f(a)} \|$ if $\mathrm{x} \in \partial B$.

Then $g(\mathrm{x}) > 0$ for each x in ∂B because $\mathrm{f(x)} \neq \mathrm{f(a)}$ if $\mathrm{x} \in \partial B$. Also, g is continuous on ∂B since f is continuous on $\bar{B}$. Since ∂B is compact, g takes on its absolute minimum (call it m) somewhere on ∂B. Note that $m > 0$ since g is positive on ∂B. Let T denote the n-ball

$$T = B\left(\mathrm{f(a)}; \frac{m}{2}\right)$$

We will prove that $T \subseteq \mathrm{f}(B)$ and this will prove the theorem. (See Fig. 5.1.)

To do this we show that $y \in T$ implies $y \in f(B)$. Choose a point y in T, keep y fixed, and define a new real-valued function h on $\bar{B}$ as follows:

$h(\mathrm{x}) = \| \mathrm{f(x)} - \mathrm{y} \|$ if $\mathrm{x} \in \bar{B}$.

Then h is continuous on the compact set $\bar{B}$ and hence attains its absolute minimum on $\bar{B}$. We will show that h attains its minimum somewhere in the open n-ball B. At the center we have $h(\mathrm{a}) = \|$

$f(a) - y \| < m/2$ since $y \in T$. Hence the minimum value of h in $\vec{B}$ must also be $< m/2$. But at each point x on the boundary ∂B we have

$$h(x) = \| f(x) - y \| = \| f(x) - f(a) - (y - f(a)) \|$$
$$\geq \| f(x) - f(a) \| - \| f(a) - y \| > g(x) - \frac{m}{2} \geq \frac{m}{2},$$

so the minimum of h cannot occur on the boundary ∂B. Hence there is an interior point c in B at which h attains its minimum. At this point the square of h also has a minimum, Since

$$h^2(x) = \| f(x) - y \|^2 = \sum_{r=1}^{n} [f_r(x) - y_r]^2,$$

and since each partial derivative $D_k(h^2)$ must be zero at c, we must have

$$\sum_{r=1}^{n} [f_r(c) - y_r] D_a f_r(c) = 0 \quad \text{for } k = 1,2, \ldots, n.$$

But this is a system of linear equations whose determinant $J_r(c)$ is not zero, since $c \in B$. Therefore $f_r(c) = y_r$ for each r, or $f(c) = y$. That is, $y \in f(B)$. Hencc $T \subseteq f(B)$ and the proof is complete.

A function $f: S \rightarrow T$ from one metric space (S, d_S) to another (T, d_T) is called an open mapping if, for every open set A in S, the image $f(A)$ is open in T.

Theorem 3:

Let A be an open subset of R^n and assume that $I: A \rightarrow R^n$ is contimuous and has finite partial derivatives $D_j f_i$ on A. If f is one-to-one on A and if $J_r(x) \neq 0$ for each x in A, then $f(A)$ is open.

Proof:

If $b \in f(A)$, then $b = f(a)$ for some a in A. There is an n-ball $B(a; r) \subseteq A$ on which f satisfies the hypotheses of Theorem 13.2, so $f(B)$ contains an n-ball with center at b. Therefore, b is an interior point of $f(A)$, so $f(A)$ is open.

The next theorem shows that a function with continuous partial derivatives is locally one-to-one near a point where the Jacobian determinant does not vanish.

Theorem 4:

Assume that $f = (f_1, \ldots, f_n)$ has continuous partial derivatives $D_j f_i$ on an open set S in R^n, and that the Jacobian determinant $J_r(a) \neq 0$ for some point a in S. Then there is an n-ball $B(a)$ on which S is one-to-one.

Proof:

Let $Z_1, \dots, Z_n$ be n points in S and let $Z = (Z_1; \dots; Z_n)$ denote that point in R^{n2} whose first n components are the components of Z_1, whose next n components are the components of Z_2, and so on. Define a real-valued function h as follows:

$h(Z) = \det[D_j f_i(Z_i)]$.

This function is continuous at those points Z in R^{n^2} where $h(Z)$ is defined because each $D_j f_i$ is continuous on S and a determinant is a polynomial in its n^2 entries. Let Z be the special point in R^{n^2} obtained by putting $Z_1 = Z_2 = \cdots = Z_n = a$

Then $h(Z) = J_f(a) \neq 0$ and hence, by continuity, there is some n-ball $B(a)$ such that $\det[D_j f_i(Z_i)] \neq 0$ if each $Z_i \in B(a)$. We will prove that f is one-to-one on $B(a)$.

Assume the contrary. That is, assume that $f(x) = f(y)$ for some pair of points $x \neq y$ in $B(a)$. Since $B(a)$ is convex, the line segment $L(x, y) \subseteq B(a)$ and we can apply the Mean-Value Theorem to each component of f to write $0 = f_i(y) - f_i(x) = \nabla f_i(Z_l) \cdot (y - x)$ for $i = 1,2, \dots, n$, where each $Z_i \in L(x, y)$ and hence $Z_i \in B(a)$. (The Mean-Value Theorem is applicable because f is differentiable on S.) But this is a system of linear equations of the form

$$\sum_{k=1}^{n} (y_k - x_k) a_{ik} = 0 \text{ with } a_{ik} = D_k f_i(Z_i)$$

The determinant of this system is not zero, since $Z_i \in B(a)$. Hence $y_k - x_k = 0$ for each k, and this contradicts the assumption that $x \neq y$. We have shown, therefore, that $x \neq y$ implies $f(x) \neq f(y)$ and hence that f is one-to-one on $B(a)$.

Note:

The reader should be cautioned that Theorem 13.4 is a local theorem and not a global theorem. The non-vanishing of $J_f(a)$ guarantees that f is one-to-one on a neighborhood of a. It does not follow that f is one-to-one on S, even when $J_t(x) \neq 0$ for every x in S. The following example illustrates this point. Let f be the complex-valued function defined by $f(z) = e^z$ if $z \in C$. If $z = x + iy$ we have

$J_f(z) = |f'(z)|^2 = |e^z|^2 = e^{2x}$.

Thus $J_f(z) \neq 0$ for every z in C. However, f is not one-to-one on C because $f(z_1) = f(z_2)$ for every pair of points z_1 and z_2 which differ by $2\pi i$.

The next theorem gives a global property of functions with nonzero Jacobian determinant.

Theorem 5:

Let A be an open subset of R^n and assume that $\mathrm{f}: A \to \mathrm{R}^n$ has continuous partial derivatives $D_j f_i$ on A. If $J_{\mathrm{f}}(\mathrm{x}) \neq 0$ for all x in A, then f is an open mapping.

Proof:

Let S be any open subset of A. If $\mathrm{x} \in S$ there is an n-ball $B(\mathrm{x})$ in which f is one-to-one (by Theorem 13.4). Therefore, by Theorem 13.3, the image $f(B(x))$ is open in R^n. But we can write $S = \bigcup_{\mathrm{x}ES} B(\mathrm{x})$. Applying f we find $\mathrm{f}(S) = \bigcup_{x \in S} \mathrm{f}(B(\mathrm{x}))$, so $\mathrm{f}(S)$ is open.

NOTE. If a function $\mathrm{f} = (f_1, \dots, f_n)$ has continuous partial derivatives on a set S, we say that f is continuously differentiable on S, and we write $f \in C'$ on S.

Theorem 4 shows that a continuously differentiable function with a nonvanishing Jacobian at a point a has a local inverse in a neighborhood of a. The next theorem gives some local differentiability properties of this local inverse function.

5.3 The Inverse Function Theorem:

Theorem 6:

Assume $\mathrm{f} = (f_1, \dots, f_n) \in C'$ on an open set S in R^n, and let $T = \mathrm{f}(S)$. If the Jacobian determinant $J_{\mathrm{r}}(\mathrm{a}) \neq 0$ for some point a in S, then there are two open sets $X \subseteq S$ and $Y \subseteq T$ and a uniquely determined function g such that

a) $a \in X$ and $\mathrm{f}(\mathrm{a}) \in Y$,

b) $Y = \mathrm{f}(X)$,

c) f is one-to-one on X,

d) g is defined on $Y, g(Y) = X$, and $g[f(x)] = \mathrm{x}$ for every x in X,

e) $g \in C'$ on Y.

Proof:

The function J_q is continuous on S and, since $J_r(a) \neq 0$, there is an n-ball $B_1(\mathrm{a})$ such that $J_1(\mathrm{x}) \neq 0$ for all x in $B_1(\mathrm{a})$. By Theorem 4, there is an n-ball $B(a) \subseteq B_1(\mathrm{a})$ on which f is one-to-one. Let B be an n-ball with center at a and radius smaller than that of $B(a)$. Then, by Theorem 2, $f(B)$

contains an n-ball with center at (a). Denote this by Y and let $X = f^{-1}(Y) \cap B$. Then X is open since both $f^{-1}(Y)$ and B are open. (See Fig. 5.2.)

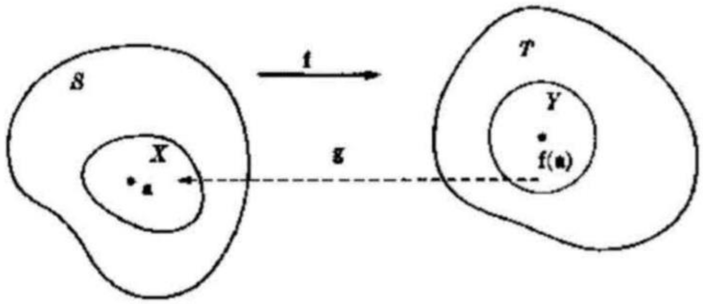

Figure 5.2

The set $\vec{B}$ (the closure of B) is compact and f is one-to-one and continuous on B. there exists a function g defined on $f(\bar{B})$ such that $g[f(x)] = x$ for all x in $\bar{B}$. Moreover, g is continuous on f($\bar{B}$). Since $X \subseteq \bar{B}$ and $Y \subseteq$ f($\bar{B}$), this proves parts (a), (b), (c) and (d). The uniqueness of g follows from (d).

Next we prove (e). For this purpose, define a real-valued function h by the equation $h(\mathrm{Z}) = \det[D_j f_i(\mathrm{Z}_i)]$, where $\mathrm{Z}_1, \dots, \mathrm{Z}_n$ are n points in S, and $\mathrm{Z} = (\mathrm{Z}_1; \dots; \mathrm{Z}_n)$ is the corresponding point in R^{n^2}. Then, arguing as in the proof of Theorem 13.4, there is an n-ball $B_2(a)$ such that $h(\mathrm{Z}) \neq 0$ if each $Z_i \in B_2$(a). We can now assume that, in the earlier part of the proof, the n-ball B (a) was chosen so that B(a) $\subseteq B_2$(a). Then $\bar{B} \subseteq B_2$(a) and $h(\mathrm{Z}) \neq 0$ if each $\mathrm{Z}_i \in \bar{B}$.

To prove (e), write $g = (g_1, \dots, g_n)$. We will show that each $g_k \in C'$ on Y. To prove that $D_r g_\mathrm{k}$ exists on Y, assume $\mathrm{y} \in Y$ and consider the difference quotient $[g_k(\mathrm{y} + t\mathrm{u}_r) - g_k(\mathrm{y})]/t$, where $\mathrm{u_r}$ is the r th unit coordinate vector. (Since Y is open, $y + tu_r \in Y$ if t is sufficiently small.) Let $\mathrm{x} = \mathrm{g(y)}$ and let $\mathrm{x}' = \mathrm{g}(\mathrm{y} + t\mathrm{u}_r)$. Then both x and x' are in X and $f(x') - f(x) = t$l. Hence $f_i(x') - f_i(\mathrm{x})$ is 0 if $i \neq r$, and is t if $i = r$. By the Mean-Value Theorem we have

$$\frac{f_i(\mathrm{x}') - f_i(\mathrm{x})}{t} = \nabla f_i(\mathrm{Z}_1) \cdot \frac{\mathrm{x}' - \mathrm{x}}{t} \quad \text{for } i = 1,2, \dots, n,$$

where each Z_f is on the line segment joining x and x'; hence $Z_i \in B$. The expression on the left is 1 or 0 , according to whether $i = r$ or $i \neq r$. This is a system of n linear equations in n unknowns $(x_j' - x_j)/t$ and has a unique solution, since

$\det[D_j f_t(\mathrm{Z}_i)] = h(\mathrm{Z}) \neq 0.$

Solving for the k th unknown by Cramer's rule, we obtain an expression for $[g_k(\mathrm{y}+\mathrm{z}_y)-g_k(\mathrm{y})]/t$ as a quotient of determinants. As $x \to 0$, the point $\mathrm{x} \to \mathrm{x}$, since g is continuous, and hence each $\mathrm{Z}_t \to \mathrm{x}$, since Z_i is on the segment joining x to x′. The determinant which appears in the denominator has for its limit the number det $[D_j f_f(x)] = J_s(\mathrm{x})$, and this is nonzero, since $\mathrm{x} \in X$. Therefore, the following limit exists:

$$\lim_{t\to 0} \frac{g_k(\mathrm{y}+tu_t) - g_k(\mathrm{y})}{t} = D_r g_k(\mathrm{y})$$

This establishes the existence of $D_r g_k(\mathrm{y})$ for each y in Y and each $r = 1,2,\dots,n$. Moreover, this limit is a quotient of two determinants involving the derivatives $D_j f_i(\mathrm{x})$. Continuity of the $D_j f_i$ implies continuity of each partial $D_r g_k$. This completes the proof of (e).

Note:

The foregoing proof also provides a method for computing $D_r g_k(y)$. In practice, the derivatives $D_r g_k$ can be obtained more easily (without recourse to a limiting process) by using the fact that, if $y = f(x)$, the product of the two Jacobian matrices Df(x) and Dg(y) is the identity matrix. When this is written out in detail it gives the following system of n^2 equations:

$$\sum_{k=1}^{n} D_k g_i(\mathrm{y}) D_j f_k(\mathrm{x}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

For each fixed i, we obtain n linear equations as j runs through the values $1,2,\dots,n$. These can then be solved for the n unknowns, $D_1 g_i(\mathrm{y}),\dots,D_n g(y)$, by Cramer's rule, or by some other method.

5.4 The Implicit Function Theorem:

A point (x_0, y_0) such that $F(x_0, y_0) = 0$, under certain conditions there will be a neighborhood of (x_0, y_0) such that in this neighborhood the relation defined by $F(x, y) = 0$ is also a function. The conditions are that F and $D_2 F$ be continuous in some neighborhood of (x_0, y_0) and that $D_2 F(x_0, y_0) \neq 0$. In its more general form, the theorem treats, instead of one equation in two variables, a system of n equations in $n + k$ variables: $f_r(x_1, \dots, x_n; t_1, \dots, t_k) = 0$ $(r = 1,2,\dots,n)$.

This system can be solved for $x_1, \dots, x_n$ in terms of $t_1, \dots, t_k$, provided that certain partial derivatives are continuous and provided that the $n \times n$ Jacobian determinant $\partial(f_1, \dots, f_n)/\partial(x_1, \dots, x_n)$ is not zero.

For brevity, we shall adopt the following notation in this theorem: Points in $(n+k)$-dimensional space R^{n+t} will be written in the form $(\mathrm{x}; \mathrm{t})$,

where $\mathrm{x} = (x_1, \dots, x_n) \in \mathrm{R}^n$ and $\mathrm{t} = (t_1, \dots, t_k) \in \mathrm{R}^k$.

Theorem 7 (Implicit function theorem):

Let $\mathrm{f} = (f_1, \dots, f_n)$ be a vector-valued function defined on an open set S in R^{n+k} with values in R^n. Suppose $\mathrm{f} \in C'$ on S. Let $(\mathrm{x}_0; \mathrm{t}_0)$ be a point in S for which $\mathrm{f}(\mathrm{x}_0; \mathrm{t}_0) = 0$ and for which the $n \times n$ determinant $\det[D_f f_l(\mathrm{x}_0; \mathrm{t}_0)] \neq 0$. Then there exists a k-dimensional open set T_0 containing t_0 and one, and only one, vector-valued function g, defined on T_0 and having values in R^4, such that

a) $\mathrm{g} \in C'$ on T_0,

b) $\mathrm{g}(\mathrm{t}_0) = \mathrm{x}_0$,

c) $\mathrm{f}(\mathrm{g}(\mathrm{t}); \mathrm{t}) = 0$ for every t in T_0.

Proof:

We shall apply the inverse function theorem to a certain vector-valued function $\mathrm{F} = (F_1, \dots, F_n; F_{n+1}, \dots, F_{n+k})$ defined on S and having values in R^{x+k}. The function F is defined as follows: For $1 \le m \le n$, let $F_m(\mathrm{x}; \mathrm{t}) = f_m(\mathrm{x}; \mathrm{t})$, and for $\mathrm{I} \le m \le k$, let $F_{n+m}(\mathrm{x}; \mathrm{t}) = t_m$. We can then write $\mathrm{F} = (\mathrm{f}; \mathrm{I})$, where $\mathrm{f} = (f_1, \dots, f_n)$ and where I is the identity function defined by $\mathrm{I}(\mathrm{t}) = \mathrm{t}$ for each t in R^{k}. The Jacobian $J_{\mathrm{F}}(\mathrm{x}; \mathrm{t})$ then has the same value as the $n \times n$ determinant $\det[D_j f_i(\mathrm{x}; \mathrm{t})]$ because the terms which appear in the last k rows and also in the last k columns of $J_{\mathrm{F}}(\mathrm{x}; \mathrm{t})$ form a $k \times k$ determinant with ones along the main diagonal and zeros elsewhere; the intersection of the first n rows and n columns consists of the determinant det $[D_j f_i(\mathrm{x}; t)]$, and $D_i F_{n+j}(\mathrm{x}; \mathrm{t}) = 0$ for $1 \le i \le n,\ 1 \le j \le k$.

Hence the Jacobian $J_{\mathrm{F}}(\mathrm{x}_0; \mathrm{t}_0) \neq 0$. Also, $\mathrm{F}(\mathrm{x}_0; \mathrm{t}_0) = (0; \mathrm{t}_0)$. Therefore, by Theorem 6, there exist open sets X and Y containing $(\mathrm{x}_0; \mathrm{t}_0)$ and $(0; \mathrm{t}_0)$, respectively, such that F is one-to-one on X, and $X = \mathrm{F}^{-1}(Y)$. Also, there exists

a local inverse function G, defined on Y and having values in X, such that $\mathrm{G}[\mathrm{F}(\mathrm{x}; \mathrm{t})] = (\mathrm{x}; \mathrm{t})$

and such that $G \in C'$ on Y.

Now G can be reduced to components as follows: $G = (v; w)$ where $v = (v_1, \ldots, v_n)$ is a vector-valued function defined on Y with values in R^n and $w = (w_1, \ldots, w_k)$ is also defined on Y but has values in R^k. We can now determine v and w explicitly. The equation $G[F(x;t)] = (x;t)$, when written in terms of the components v and w, gives us the two equations

$v[F(x;t)] = x$ and $w[F(x;t)] = t$.

But now, every point $(x;t)$ in Y can be written uniquely in the form $(x;t) = F(x';t')$ for some $(x';t')$ in X, because F is one-to-one on X and the inverse image $F^{-1}(Y)$ contains X. Furthermore, by the manner in which F was defined, when we write $(x;t) = F(x';t')$, we must have $t' = t$. Therefore, $v(x;t) = v[F(x';t)] = x'$ and $w(x;t) = w[F(x';t)] = t$

Hence the function G can be described as follows: Given a point $(x;t)$ in Y, we have $G(x;t) = (x';t)$, where x' is that point in R^* such that $(x;t) = F(x';t)$. This statement implies that

$F[v(x;t);t] = (x;t)$ for every $(x;t)$ in Y.

Now we are ready to define the set T_0 and the function g in the theorem.

Let $T_0 = \{t: t \in R^k,\ (0;t) \in Y\}$

and for each t in T_0 define $g(t) = v(0;t)$. The set T_0 is open in R^k. Moreover, $g \in C'$ on T_0 because $G \in C'$ on Y and the components of g are taken from the components of G. Also,

$g(t_0) = v(0;t_0) = x_0$

because $(0;t_0) = F(x_0;t_0)$. Finally, the equation $F[v(x;t);t] = (x;t)$, which holds for every $(x;t)$ in Y, yields (by considering the components in R^q) the equation $f[v(x;t);t] = x$. Taking $x = 0$, we see that for every t in T_0, we have $f[g(t);t] = 0$, and this completes the proof of statements (a), (b), and (c). It remains to prove that there is only one such function g. But this follows at once from the one-to-one character of f. If there were another function, say h, which satisfied (c), then we would have $f[g(t);t] = f[h(t);t]$, and this would imply $(g(t);t) = (h(t);t)$, or $g(t) = h(t)$ for every t in T_0.

5.5. Extrema of Real-Valued Functions of One Variable:

In the remainder of this chapter we shall consider real-valued functions f with a view toward determining those points (if any) at which f has a local extremum, that is, either a local maximum or a local minimum.

We have already obtained one result in this connection for functions of one variable. In that theorem we found that a necessary condition for a function f to have a local extremum at an interior point c of an interval is that $f'(c) = 0$, provided that $f'(c)$ exists. This condition, however, is not sufficient, as we can see by taking $f(x) = x^3, c = 0$. We now derive a sufficient condition.

Theorem 8:

For some integer $n \geq 1$, let f have a continuous nth derivative in the open interval (a, b). Suppose also that for some interior point c in (a, b) we have

$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0,$ but $f^{(n)}(c) \neq 0$

Then for n even, f has a local minimum at c if $f^{(n)}(c) > 0$, and a local maximum at c if $f^{(n)}(c) < 0$. If n is odd, there is neither a local maximum nor a local minimum at c.

Proof:

Since $f^{(n)}(c) \neq 0$, there exists an interval $B(c)$ such that for every x in $B(c)$, the derivative $f^{(n)}(x)$ will have the same sign as $f^{(n)}(c)$. Now by Taylor's formula (Theorem 5.19), for every x in $B(c)$ we have $f(x) - f(c) = \frac{f^{(n)}(x_1)}{n!}(x - c)^n$, where $x_1 \in B(c)$.

If n is even, this equation implies $f(x) \geq f(c)$ when $f^{(n)}(c) > 0$, and $f(x) \leq f(c)$ when $f^{(k)}(c) \leq 0$. If n is odd and $f^{(n)}(c) > 0$, then $f(x) > f(c)$ when $x > c$, but $f(x) < f(c)$ when $x < c$, and there can be no extremum at c. A similar statement holds if n is odd and $f^{(n)}(c) < 0$. This proves the theorem.

5.6 Extrema of Real-Valued Functions of Several Variables:

The condition is that each partial derivative $D_k f$(a) must be zero at that point. We can also state this in terms of directional derivatives by saying that $f'(s;$ u$)$ must be zero for every direction u. The converse of this statement is not true, however. Consider the following example of a function of two real variables: $f(x, y) = (y - x^2)(y - 2x^2)$.

Here we have $D_1 f(0,0) = D_2 f(0,0) = 0$. Now $f(0,0) = 0$, but the function assumes both positive and negative values in every neighborhood of (0,0), so there is neither a local maximum nor a local minimum at (0,0). (See Fig. 5.3.)

This example illustrates another interesting phenomenon. If we take a fixed straight line through the origin and restrict the point (x, y) to move along this line toward (0,0), then the point will finally enter the region above the parabola $y = 2x^2$ (or below the parabola $y = x^2$) in which $f(x, y)$ becomes and stays positive for every $(x, y) \neq (0,0)$. Therefore, along every such line, f has a minimum at (0,0), but the origin is not a local minimum in any two-dimensional neighborhood of (0,0).

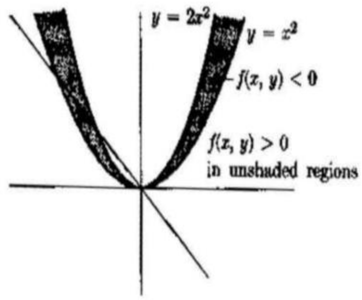

Figure 5.3

Definition 9:

If f is differentiable at a and if $\nabla f(\mathrm{a}) = 0$, the point a is called a stationary point of f. A stationary point is called a saddle point if every n-ball $B(\mathrm{a})$ contains points x such that $f(\mathrm{x}) > f(\mathrm{a})$ and other points such that $f(\mathrm{x}) < f(\mathrm{a})$.

In therefore going example, the origin is a saddle point of the function.

To determine whether a function of n variables has a local maximum, a local minimum, or a saddle point at a stationary point a, we must determine the algebraic sign of $f(\mathrm{x}) - f(\mathrm{a})$ for all x in a neighborhood of a. As in the one-dimensional case, this is done with the help of Taylor's formula (Chapter 4, Theorem 14). Take $m = 2$ and $\mathrm{y} = \mathrm{a} + \mathrm{t}$ in (chapter 4, theorem 14.) If the partial derivatives of f are differentiable on an n-balt $B(a)$ then

$$f(\mathrm{a} + \mathrm{t}) - f(\mathrm{a}) = \nabla f(\mathrm{a}) \cdot \mathrm{t} + \frac{1}{2} f''(\mathrm{z}; \mathrm{t}), \quad \ldots\ldots\ldots\ldots (3)$$

where z lies on the line segment joining a and a + t, and

$$f''(\mathrm{z}; \mathrm{t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(\mathrm{z}) \mathrm{t}_i t_j$$

At a stationary point we have $\nabla f(\mathrm{a}) = 0$ so equation (3) becomes

$f(\mathrm{a}+\mathrm{t})-f(\mathrm{a})=\frac{1}{2} f''(\mathrm{z};\mathrm{t})$.

Therefore, as $\mathrm{a}+\mathrm{t}$ ranges over $B(\mathrm{a})$, the algebraic sign of $f(\mathrm{a}+\mathrm{t})-f(\mathrm{a})$ is determined by that of $f''(\mathrm{z};\mathrm{t})$. We can write (3) in the form $f(\mathrm{a}+\mathrm{t})-f(\mathrm{a})=\frac{1}{2} f''(\mathrm{a};\mathrm{t})+\| \mathrm{t} \|^2 E(\mathrm{t})$, ...(4)

Where, The inequality $\| \mathrm{t} \|^2 E(\mathrm{t})=\frac{1}{2} f''(\mathrm{z};\mathrm{t})-\frac{1}{2} f''(\mathrm{a};\mathrm{t})$.

$$\| \mathrm{t} \|^2 |E(\mathrm{t})| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{i=1}^{n} \left|D_{i,j} f(\mathrm{z})-D_{i,j} f(\mathrm{a})\right| \| \mathrm{t} \|^2,$$

shows that $E(\mathrm{t}) \rightarrow 0$ as $\mathrm{t} \rightarrow 0$ if the second-order partial derivatives of f are continuous at a. Since $\| \mathrm{t} \|^2 E(t)$ tends to zero faster than $\|t_1\|^2$, it seems reasonable to expect that the algebraic sign of $f(\mathrm{a}+\mathrm{t})-f(\mathrm{a})$ should be determined by that of $f''(\mathrm{a};\mathrm{t})$. This is what is proved in the next theorem.

Theorem 10 (Second-derivative test for extrema):

Assume that the second-order partial derivatives $D_{i,j} f$ exist in an n-ball $B(\mathrm{a})$ and are continuous at a, where a is a stationary point of f. Let $Q(\mathrm{t})=\frac{1}{2} f''(\mathrm{a};\mathrm{t})=\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(\mathrm{a}) t_i t_j$

a) If $Q(\mathrm{t})>0$ for all $\mathrm{t} \neq 0, f$ has a relative minimum at a.

b) If $Q(\mathrm{t})<0$ for all $\mathrm{t} \neq 0$, f has a relative maximum at a.

c) If $Q(\mathrm{t})$ takes both positive and negative values, then f has a saddle point at a.

Proof:

The function Q is continuous at each point t in R^n. Let $S=\{t: \| t \|=1\}$ denote the boundary of the n-ball $B(0;1)$. If $Q(t)>0$ for all $t \neq 0$, then $Q(t)$ is positive on S. Since S is compact, Q has a minimum on S (call it m), and $m>0$. Now $Q(ct)=c^2 Q(t)$ for every real c. Taking $c=1/\| \mathrm{t} \|$ where $t \neq 0$ we see that ct $\in S$ and hence $c^2 Q(t) \geq m$, so $Q(t) \geq m \| t \|^2$. Using this in (4) we find $f(\mathrm{a}+\mathrm{t})-f(\mathrm{a})=Q(\mathrm{t})+\| \mathrm{t} \|^2 E(\mathrm{t}) \geq m \| \mathrm{t} \|^2+\| \mathrm{t} \|^2 E(\mathrm{t})$.

Since $E(\mathrm{t}) \rightarrow 0$ as $\mathrm{t} \rightarrow 0$, there is a positive number r such that $|E(\mathrm{t})|<\frac{1}{2} m$ whenever $0<\| t \|<r$. For such t we have $0 \leq \| t \|^2 |E(t)|<\frac{1}{2} m \| t \|^2$, so

$$f(\mathrm{a}+\mathrm{t})-f(\mathrm{a})>m \| \mathrm{t} \|^2-\frac{1}{2} m \| \mathrm{t} \|^2=\frac{1}{2} m \| t \|^2>0.$$

Therefore f has a relative minimum at a, which proves (a). To prove (b) we use a similar argument, or simply apply part (a) to $-f$.

Finally, we prove (c). For each $\lambda > 0$ we have, from (4),

$f(\mathrm{a}+\lambda t) - f(\mathrm{a}) = Q(\lambda t) + \lambda^2 \parallel t \parallel^2 E(\lambda t) = \lambda^2\{Q(t) + \parallel \mathrm{t} \parallel^2 E(\lambda t)\}$.

Suppose $Q(t) \neq 0$ for some t. Since $E(\mathrm{y}) \to 0$ as $\mathrm{y} \to 0$, there is a positive r such that

$\parallel \mathrm{t} \parallel^2 E(\lambda t) < \frac{1}{2}|Q(\mathrm{t})|$ if $0 < \lambda < r$.

Therefore, for each such λ the quantity $\lambda^2\{Q(t) + \parallel t \parallel^2 E(\lambda t)\}$ has the same sign as $Q(\mathrm{t})$. Therefore, if $0 < \lambda < r$, the difference $f(\mathrm{a}+\lambda\mathrm{t}) - f(\mathrm{a})$ has the same sign as $Q(\mathrm{t})$. Hence, if $Q(\mathrm{t})$ takes both positive and negative values, it follows that f has a saddle point at a.

Note:

A real-valued function Q defined on R^x by an equation of the type

$$Q(\mathrm{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

where $\mathrm{X} = (x_1, \dots, x_n)$ and the a_{ij} are real is called a quadratic form. The form is called symmetric if $a_{ij} = a_j$ for all i and j, positive definite if $\mathrm{x} \neq 0$ impties $Q(\mathrm{x}) > 0$, and negative definite if $\mathrm{x} \neq 0$ implies $Q(\mathrm{x}) < 0$.

In general, it is not easy to determine whether a quadratic form is positive or negative definite. One criterion, involving eigenvalues, is described in Reference

5.1, Another, involving determinants, can be described as follows. Let $\Delta = \det[a_{ij}]$ and let Δ_k denote the determinant of the $k \times k$ matrix obtained by deleting the last $(n-k)$ rows and columns of $[a_{ij}]$. Also, put $\Delta_0 = 1$. From the theory of quadratic forms it is known that a necessary and sufficient condition for a symmetric form to be positive definite is that the $n+1$ numbers $\Delta_0, \Delta_f, \dots, \Delta_n$ be positive. The form is negative definite if, and only if, the same $n+1$ numbers are alternately positive and negative. The quadratic form which appears in (5) is symmetric because the mixed partials $D_{i,j}f(\mathrm{a})$ and $D_{j.j}f(\mathrm{a})$ are equal. Therefore, under the conditions of Theorem 13.10, we see that f has a local minimum at a if the $(n+1)$ numbers $\Delta_0, \Delta_1, \dots, \Delta_n$ are all positive, and a local maximum if these numbers are alternately positive and negative. The case $n = 2$ can be handled directly and gives the following criterion.

Theorem 11:

Let f be a real-valued function with continuous second-order partial derivatives at a stationary point a in R^2. Let

$A = D_{1,1}f(\mathrm{a}),\ B = D_{1,2}f(\mathrm{a}),\ C = D_{2,2}f(\mathrm{a}),$

and let $\Delta = \det\begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2$

Then we have:

a) If $\Delta > 0$ and $A > 0$, f has a relative minimum at a.

b) If $\Delta > 0$ and $A < 0$, f has a relative maximum at a.

c) If $\Delta < 0, f$ has a saddle point at a.

Proof:

In the two-dimensional case we can write the quadratic form in (5) as follows:

$$Q(x,y) = \frac{1}{2}\{Ax^2 + 2Bxy + Cy^2\}.$$

If $A \neq 0$, this can also be written as

$$Q(x,y) = \frac{1}{2A}\{(Ax + By)^2 + \Delta y^2\}.$$

If $\Delta > 0$, the expression in brackets is the sum of two squares, so $Q(x,y)$ has the same sign as A. Therefore, statements (a) and (b) follow at once from parts (a) and (b) of Theorem 13.10.

If $\Delta < 0$, the quadratic form is the product of two linear factors. Therefore, the set points (x,y) such that $Q(x,y) = 0$ consists of two lines in the xy-plane intersecting at (0,0). These lines divide the plane into four regions; $Q(x,y)$ is positive in two of these regions and negative in the other two. Therefore f has a saddle point at a.

Note: If $\Delta = 0$, there may be a local maximum, a local minimum, or a saddle point at a.

13.7 Extremum problems with side conditions:

Consider the following type of extremum problem. Suppose that $f(x,y,z)$ represents the temperature at the point (x,y,z) in space and we ask for the maximum or minimum value of the temperature on a certain surface. If the equation of the surface is given explicitly in the form $\dot{z} = h(x,y)$, then in the expression $f(x,y,z)$ we can replace z by $h(x,y)$ to obtain the temperature on the surface as a function of x and y alone, say $F(x,y) = f[x,y,h(x,y)]$. The problem is then reduced to finding the extreme values of F. However, in practice, certain difficulties arise. The equation of the surface might be given in an implicit form, say $g(x,y,z)=0$, and it may be impossible, in practice, to solve this equation explicitly for z in terms of x and y, or even for x or y in terms of the remaining variables. The problem might be further complicated by asking for the extreme values of the temperature at those points which lie on a given curve in space. Such a curve is the intersection of two surfaces, say $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$. If we could solve these two equations simultaneously, say for x and y in terms of z, then we could introduce these expressions into f and obtain a new function of z alone, whose extrema we would then seek. In general, however, this procedure cannot be carried out and a more practicable method must be sought. A very elegant and useful method for attacking such problems was developed by Lagrange. Lagrange's method provides a necessary condition for an extremum and can be described as follows. Let $f(x_1,\dots,x_n)$ be an expression whose extreme values are sought when the variables are restricted by a certain number of side conditions, say $g_1(x_1,\dots,x_n)=0,\dots,g_m(x_1,\dots,x_n)=0$. We then form the linear combination

$$\phi(x_1,\dots,x_n)=f(x_1,\dots,x_n)+\lambda_1 g_1(x_1,\dots,x_n)+\cdots+\lambda_m g_m(x_1,\dots,x_n),$$

where $\lambda_1,\dots,\lambda_m$ are m constants. We then differentiate ϕ with respect to each coordinate and consider the following system of $n+m$ equations:

$$D_r\phi(x_1,\dots,x_n)=0, \qquad r=1,2,\dots,n,$$
$$g_k(x_1,\dots,x_n)=0, \qquad k=1,2,\dots,m.$$

Lagrange discovered that if the point $(x_1,\dots,x_n)$ is a solution of the extremum problem, then it will also satisfy this system of $n+m$ equations. In practice, one attempts to solve this system for the $n+m$ "unknowns," $\lambda_1,\dots,\lambda_m$, and $x_1,\dots,x_n$. The points $(x_1,\dots,x_n)$ so obtained must then be tested to determine whether they yield a maximum, a minimum, or neither. The numbers $\lambda_1,\dots,\lambda_m$, which are introduced only to help solve the system for $x_1,\dots,x_n$, are known as Lagrange's multipliers. One multiplier is introduced for each side condition.

A complicated analytic criterion exists for distinguishing between maxima and minima in such problems. However, this criterion is not very useful in practice and in any particular problem it is usually easier to rely on some other means (for example, physical or geometrical considerations) to make this distinction.

The following theorem establishes the validity of Lagrange's method:

Theorem 12:

Let f be a real-valued function such that $f \in C'$ on an open set S in R^*. Let $g_1, \dots, g_m$ be m real-valued functions such that $g = (g_1, \dots, g_m) \in C'$ on S, and assume that $m < n$. Let X_0 be that subset of S on which g vanishes, that is, $X_0 = \{x: x \in S, g(x) = 0\}$.

Assume that $x_0 \in X_0$ and assume that there exists an n-ball $B(x_0)$ such that $f(x) \leqslant f(x_0)$ for all x in $X_0 \cap B(x_0)$ or such that $f(x') \geq f(x_0)$ for all x in $X_0 \cap B(x_0)$. Assume also that the m-rowed determinant det $[D_j g_1(x_0)] \neq 0$. Then there exist m real numbers $\lambda_1, \dots, \lambda_m$ such that the following n equations are satisfied: $D_r f(x_0) + \sum_{k=1}^{m} \lambda_k D_r g_k(x_0) = 0 \; (r = 1,2, \dots, n)$ ……..(6)

Note:

The n equations in (6) are equivalent to the following vector equation:

$\nabla f(x_0) + \lambda_1 \nabla g_1(x_0) + \cdots + \lambda_m \nabla g_m(x_0) = 0$

Proof. Consider the following system of m linear equations in the m unknowns $\lambda_1, \dots, \lambda_m$:

$$\sum_{k=1}^{m} \lambda_k D_r g_k(x_0) = -D_r f(x_0) \; (r = 1,2, \dots, m)$$

This system has a unique solution since, by hypothesis, the determinant of the system is not zero. Therefore, the first m equations in (6) are satisfied. We must now verify that for this choice of $\lambda_1, \dots, \lambda_m$, the remaining $n - m$ equations in (6) are also satisfied.

To do this, we apply the implicit function theorem. Since $m < n$, every point x in S can be written in the form $x = (x'; t)$, say, where $x' \in R^m$ and $t \in R^{n-m}$ In the remainder of this proof we will write x' for $(x_1, \dots, x_m)$ and t for $(x_{m+1}, \dots, x_n)$, so that $t_k = x_{m+k}$. In terms of the vector-valued function $g = (g_1, \dots, g_m)$, we can now write

$g(x'_0; t_0) = 0$ if $x_0 = (x'_0; t_0)$.

Since $g \in C'$ on S, and since the determinant $\det[D_j g_1(x'_0; t_0)] \neq 0$, all the conditions of the implicit function theorem are satisfied. Therefore, there exists an $(n - m)$-dimensional neighborhood T_0 of t_0 and a unique vector-valued function $h = (h_1, \dots, h_m)$, defined on T_0 and having values in R^m such that $h \in C'$ on T_0, $h(t_0) = x'_0$, and for every t in T_0, we have $g[h(t); t] = 0$. This amounts to saying that the system of m equations

$g_1(x_1, \dots, x_n) = 0, \dots, g_m(x_1, \dots, x_n) = 0$

can be solved for $x_1, \dots, x_m$ in terms of $x_{m+1}, \dots, x_m$, giving the solutions in the form $x_r = h_r(x_{n+1}, \dots, x_n), r = 1,2, \dots, m$. We shall now substitute these expressions for $x_1, \dots, x_m$ into the expression $f(x_1, \dots, x_n)$ and also into each

expression $g_p(x_1, \dots, x_n)$. That is to say, we define a new function F as follows:

$F(x_{m+1}, \dots, x_n) = f[h_1(x_{m+1}, \dots, x_n), \dots, h_m(x_{m+1}, \dots, x_n); x_{m+1}, \dots, x_n]$

and we define m new functions $G_1, \dots, G_m$ as follows:

$G_p(x_{m+1}, \dots, x_n) = g_p[h_1(x_{m+1}, \dots, x_n), \dots, h_m(x_{m+1}, \dots, x_n); x_{m+1}, \dots, x_n]$.

More briefly, we can write $F(t) = f[\mathrm{H}(t)]$ and $G_p(t) = g_p[\mathrm{H}(\mathrm{t})]$, where $\mathrm{H}(t) = (h(t); t)$. Here t is restricted to lie in the set T_0.

Each function G_p so defined is identically zero on the set T_0 by the implicit function theorem. Therefore, each derivative D, G_p is also identically zero on T_0 and, in particular, $D_r G_p(\mathrm{t}_0) = 0$. But by the chain rule (chapter 4 equation.20), we can compute these derivatives as follows:

$$D_r G_p(\mathrm{t}_0) = \sum_{k=1}^{n} D_k g_p(\mathrm{x}_0) D_r H_k(\mathrm{t}_0) \ (r = 1,2, \dots, n-m).$$

But $H_k(t) = h_k(t)$ if $1 \le k \le m$, and $H_k(t) = x_k$ if $m+1 \le k \le n$. Therefore, when $m+1 \le k \le n$, we have $D_r H_k(\mathrm{t}) \equiv 0$ if $m + r \ne k$ and $D_r H_{m+r}(t) = 1$ for every t. Hence the above set of equations becomes $\sum_{k=1}^{m} D_k g_p(\mathrm{x}_0) D h_k(\mathrm{t}_0) + D_{m+r} g_p(\mathrm{x}_0) = 0 \begin{cases} p = 1,2, \dots, m, \\ r = 1,2, \dots, n-m. \end{cases}$ …….(7)

By continuity of h, there is an $(n-m)$-ball $B(t_0) \subseteq T_0$ such that $t \in B(t_0)$ implies $(\mathrm{h(t);t}) \in B(\mathrm{x}_0)$, where $B(\mathrm{x}_0)$ is the n-ball in the statement of the theorem. Hence, $t \in B(\mathrm{t}_0)$ implies $(\mathrm{h(t);t}) \in X_0 \cap B(\mathrm{x}_0)$ and therefore, by hypothesis, we have either $F(t) \le F(t_0)$ for all t in $B(t_0)$ or else we have $F(t) \ge F(t_0)$ for all t in $B(\mathrm{t}_0)$. That is, F has a local maximum or a local minimum at the interior point t_0. Each partial derivative $D_r F(t_0)$ must therefore be zero. If we use the chain rule to compute these derivatives, we find

$$D_r F(\mathrm{t}_0) = \sum_{k=1}^{n} D_k f(\mathrm{x}_0) D_r H_k(\mathrm{t}_0). \ (r = 1, \dots, n-m),$$

and hence we can write $\sum_{k=1}^{m} D_k f(\mathrm{x}_0) D h_k(\mathrm{t}_0) + D_{m+r} f(\mathrm{x}_0) = 0 \ (r = 1, \dots, n-m)$ …….(8)

If we now multiply (7) by λ_p, sum on p, and add the result to (8), we find

$$\sum_{k=1}^{m}\left[D_k f(\mathrm{x}_0)+\sum_{p=1}^{m} \lambda_p D_k g_p(\mathrm{x}_0)\right] D_r h_k(\mathrm{t}_0)+D_{m+r} f(\mathrm{x}_0)+\sum_{p=1}^{m} \lambda_p D_{m+r} g_p(\mathrm{x}_0)=0,$$

for $r=1, \ldots, n-m$. In the sum over k, the expression in square brackets vanishes because of the way $\lambda_1, \ldots, \lambda_m$ were defined. Thus we are left with

$$D_{m+r} f(\mathrm{x}_0)+\sum_{p=1}^{m} \lambda_p D_{m+r} g_p(\mathrm{x}_0)=0 \ (r=1,2, \ldots, n-m).$$

Example:

A quadric surface with center at the origin has the equation

$Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy=1$. Find the lengths of its semi-axes.

Solution:

Let us write (x_1, x_2, x_3) instead of (x, y, z), and introduce the quadratic form

$q(x)=\sum_{j=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j$ .... (9)

where $\mathrm{x}=(x_1, x_2, x_3)$ and the $a_{ij}=a_{ji}$ are chosen so that the equation of the surface becomes $q(x)=1$. (Hence the quadratic form is symmetric and positive definite.) The problem is equivalent to finding the extreme values of $f(\mathrm{x})=\| \mathrm{x} \|^2=x_1^2+x_2^2+x_3^2$ subject to the side condition $g(\mathrm{x})=0$, where $g(\mathrm{x})=q(\mathrm{x})-1$. Using Lagrange's method, we introduce one multiplier and consider the vector equation $\nabla f(\mathrm{x})+\lambda \nabla q(\mathrm{x})=0$ ....(10)

(since $\nabla g=\nabla q$). In this particular case, both f and q are homogeneous functions of degree 2 and $\mathrm{x} \cdot \nabla f(\mathrm{x})+\lambda \mathrm{x} \cdot \nabla q(\mathrm{x})=2 f(\mathrm{x})+2 \lambda q(\mathrm{x})=0$.

Since $q(\mathrm{x})=1$ on the surface we find $\lambda=-f(\mathrm{x})$, and equation (10)

becomes $t \nabla f(\mathrm{x})-\nabla q(\mathrm{x})=0$, ...(11)

where $t=1 / f(\mathrm{x})$. (We cannot have $f(\mathrm{x})=0$ in this problern.) The vector equation (11) then leads to the following three equations for x_1, x_2, x_3 :

$$(a_{11}-t) x_1+a_{12} x_2+a_{13} x_3=0$$
$$a_{21} x_1+(a_{22}-t) x_2+a_{23} x_3=0$$
$$a_{31} x_1+a_{32} x_2+(a_{33}-t) x_3=0.$$

Since $\mathrm{x}=0$ cannot yield a solution to our problem, the determinant of this systern must vanish. That is, we must have $\left|\begin{array}{ccc}a_{11}-t & a_{12} & a_{13} \\ a_{21} & a_{22}-t & a_{23} \\ a_{31} & a_{32} & a_{33}-t\end{array}\right|=0.$ ....(12)

Equation (12) is called the characteristic equation of the quadratic form in (9). In this case, the geometrical nature of the problem assures us that the three roots t_1, t_2, t_3 of this cubic must be real and positive. [Since q(x) is symmetric and positive definite, the general theory of quadratic forms also guarantees that the roots of (12) are all real and positive. The semi-axes of the quadric surface are $t_1^{-1/2}, t_2^{-1/2}, t_3^{-1/2}$.

Study Learning Material Prepared by

Mrs. S. KALAISELVI M.SC., M.Phil., B.Ed., (Ph.D.),

Assistant Professor,

Department of Mathematics,

Sarah Tucker College (Autonomous),

Tirunelveli-627007.

Tamil Nadu, India.